

FULL PAPER

Spectral Bounds for the First Steklov Eigenvalue via Reilly-Type Inequalities

Prepared by

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Abstract

Abstract. In this paper, we present some results regarding Reilly's inequality for the first Steklov eigenvalue. Consider a compact submanifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. We will address the Steklov problem, specifically when $\mathcal{L}(\partial\mathcal{M})$ represents the length of $\partial\mathcal{M}$.

Keywords : Steklov eigenvalue ; Reilly inequality ; Spectral theory ; Variational methods ; Riemannian manifolds ; Steklov problem ; Euclidean space.

1. Introduction

The mean curvature of an m -dimensional Riemannian manifold $(\mathcal{M}, g)$ is represented as $\mathcal{H}$. The Euclidean space $(\mathbb{R}^n, can)$ supports a metric-preserving immersion map ψ in this manifold. A classical result due to Reilly (see [2]) establishes an extrinsic estimate for the first positive eigenvalue $\lambda_1(\mathcal{M})$ of the Laplace–Beltrami operator $\Delta = -tr \circ \text{Hess}$ on $(\mathcal{M}, g)$, expressed via the mean curvature associated with an isometric immersion of $\mathcal{M}$. When $\mathcal{M}$ is compact and boundaryless.

$$\psi: \lambda_1 \leq \frac{m}{\text{Vol}(\mathcal{M})} \int_{\mathcal{M}} |\mathcal{H}|^2 dv_g \quad (1)$$

Here, dv_g denotes the volume element induced by the Riemannian metric g , $\text{Vol}(\mathcal{M})$ the total volume of the manifold $(\mathcal{M}, g)$, and $|\mathcal{H}|^2$ the squared norm of the mean curvature vector associated to the deep involvement ψ .

Specifically, the a broader result is demonstrated in the following Theorem if we represent the r -th higher-order mean curvature functions of the map defining the immersion by $\mathcal{H}_r$ (see [2]). In this paper, we demonstrated that If a compact Riemannian surface $(\mathcal{M}, g)$ endowed with a boundary is isometrically submerged in $\psi_j : (\mathcal{M}^m, g) \rightarrow (\mathbb{R}^n, can)$, then

$$\int_{\partial\mathcal{M}} \sum_{j=1}^n \langle \psi_j, \mathcal{H}_j \rangle dv_{g_j} = - \int_{\partial\mathcal{M}} tr(\mathcal{T}) dv_{g_j}$$

where $\mathcal{H}_{\mathcal{T}}$ is a section of the normal bundle associated with $\mathcal{T}$, and $\mathcal{T}$ is a symmetric (0,2)-tensor with zero divergence on $\partial\mathcal{M}$.

Theorem 1.1:

(i) Suppose $n > m + 1$ and $r \in \{1, 3, \dots, m\}$. Then we have

$$\lambda_1 \left(\int_{\mathcal{M}} \mathcal{H}_{r-1} dv_g \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\mathcal{M}} |\mathcal{H}_r|^2 dv_g \right) \quad (2)$$

Assume that equality is attained in the preceding inequality for some value of r , and that the associated mean curvature vector $\mathcal{H}_r$ is not identically zero. Under these conditions, the map ψ provides a minimal immersion of $\mathcal{M}$ into a certain hypersphere of $\mathbb{R}^n$, and furthermore $\mathcal{H}_r$ is parallel along the normal bundle of $\mathcal{M}$ in $\mathbb{R}^n$. In the special case $r = 1$, if equality holds, then ψ embeds $\mathcal{M}$ minimally in a hypersphere of $\mathbb{R}^n$; in this situation, no assumption is required on the non-vanishing of the mean curvature vector $\mathcal{H}_1 = \mathcal{H}$.

(ii) For $r \in \{0, 1, 2, \dots, m\}$, and suppose $n = m + 1$, then we get

$$\lambda_1 \left(\int_{\mathcal{M}} \mathcal{H}_{r-1} dv_g \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\mathcal{M}} \mathcal{H}_r^2 dv_g \right) \quad (3)$$

Equality holds in the Eq. (3) for any $r \in \{0, 1, \dots, m\}$ if and only if the immersion $\psi: \mathcal{M} \rightarrow \mathbb{R}^n$ realizes $\mathcal{M}$ as a hypersphere.

Based on a partial finding from [14], it is straightforward to extend the Reilly inequality to immersed manifolds of spheres when $r = 1$. Additionally, ([3] and [12]) have extended it to hyperbolic submanifolds. Grosjean generalized the findings for spherical and hyperbolic submanifolds to all r (see to [5] and [13]). Keep in mind that up to now, there have been no comparable inequalities where $\mathcal{M}$ has a border.

Suppose $\mathcal{M}$ be an ordered Riemannian manifold with a boundary represented as $\partial\mathcal{M}$. The current investigation addresses the initial positive spectral value p_1 of the Steklov issue.

$$\begin{cases} \nabla f = 0 & \text{in } \mathcal{M} \\ \frac{\partial f}{\partial \nu} = pf & \text{on } \partial\mathcal{M} \end{cases} \quad (4)$$

In this case, ν represents the outer component parallel of $\partial\mathcal{M}$ in $\mathcal{M}$.

We observe that, in general, the Steklov problem can be considered with an amended boundary and the density ω satisfies the situation $\frac{\partial f}{\partial \nu} = p\omega f$. However, for the sake of simplicity, we evaluate exclusively the case $\omega \equiv 1$. Minor adjustments can be made to simply treat the general case. The Steklov boundary problem Eq. (4) is known to have a discontinuous frequency.

$$0 = p_0 < p_1 \leq p_2 \leq \dots \rightarrow \infty.$$

The problem of boundaries Steklov [6] first showed Eq. (4) for a limited planar region $\mathcal{M}$.

On a manifold $\mathcal{M}$, the equilibrium temperature is modeled by a function f , with the corresponding boundary flux being proportional to the temperature. Since the Steklov problem shares both eigenvalues and eigenfunctions with the Dirichlet–Neumann operator, which

associates boundary values on $\partial\mathcal{M}$ with the normal derivatives of their harmonic extensions inside $\mathcal{M}$, it plays a fundamental role in harmonic analysis and the theory of inverse problems (see [11]). In earlier work, Weinstock [7] established the first isoperimetric inequality for the smallest non-trivial Steklov eigenvalue p_1 in the setting of a simply connected planar domain $\mathcal{M}$.

$$p_1 \leq \frac{2\pi}{\mathcal{L}(\partial\mathcal{M})}, \quad (5)$$

2. Reilly-Steklov Disparity

Let $\mathcal{L}(\partial\mathcal{M})$ be the distance of the boundary $\partial\mathcal{M}$. Subsequent to that, other writers have since addressed this problem and produced different eigenvalue estimates (see [8], [9], and [16]). Regrettably, we note that most findings on this topic are limited to connected planar regions.

The Weinstock Eq. (5) has recently been generalized to encompass compact Riemannian surfaces $\mathcal{M}$ of genus ψ possessing k connected boundary components, as demonstrated in [18].

$$p_1 \mathcal{L}(\partial\mathcal{M}) \leq 2(\psi + k)\pi$$

This inquiry parallels a previous study by Li and Yau [1] (subsequently expanded in [3]) concerning the initial property of the Laplace–Beltrami operator (or the Neumann Laplacian for manifolds with boundaries); this work also examined the Steklov boundary value problem on a compact Riemannian structure of dimension m whose boundary is non-empty that permits conformal maps that are proper into the unit ball $\mathcal{B}^n$. They set a maximum value for p_1 in relation to the conformal volume relative to dimension n $\text{Vol}_{rc}(\mathcal{M}, n)$ [18]:

$$p_1 \text{Vol}(\partial\mathcal{M}) \text{Vol}(\mathcal{M})^{\frac{2-m}{m}} \leq m \text{Vol}_{rc}(\mathcal{M}, n)^{\frac{2}{m}}$$

The focus of this evaluation will be the Steklov challenge, based on a dense Riemannian manifold $\mathcal{M}$ that is geometrically embedded in $\mathbb{R}^2$ or $\mathbb{S}^2$. Similar to the bounds for λ_1 determined by Reilly and Grosjean, we shall construct upper bounds for p_1 using the r - t h mean curvatures of $\partial\mathcal{M}$ in either $\mathbb{R}^2$ or $\mathbb{S}^2$. We establish the following for Euclidean submanifolds.

Theorem 2.1: Suppose we have a compact Riemannian manifold $(\mathcal{M}^m, g)$ with a boundary $\partial\mathcal{M}$ and an isometric immersion $\psi: (\mathcal{M}^m, g) \rightarrow (\mathbb{R}^n, can)$. Let $\mathcal{H}_r$ be the r - t h mean curvatures of $\partial\mathcal{M}$ in $\mathbb{R}^n$, leads to

(i) Suppose $n > m$ and $r \in \{1, 3, \dots, m-1\}$, then

$$p_1 \left(\int_{\partial\mathcal{M}} \mathcal{H}_{r-1} dv_g \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_r|^2 dv_g \right) \quad (6)$$

If, additionally, $\mathcal{H}_r$ does not disappear precisely, then the equality in Eq. (6) for this choice of r holds precisely when ψ essentially realizes $\mathcal{M}$ as a submanifold of $\mathcal{B}^n\left(\frac{1}{p_1}\right)$, with $\psi(\partial\mathcal{M}) \subset \mathcal{B}^n\left(\frac{1}{p_1}\right)$, perpendicularly, and $\mathcal{H}_r$ is commensurate to ψ .

In specific, when $r = 1$ in Eq. (6) holds if and only if ψ represents a minimum penetration of $\mathcal{M}$ into $\mathcal{B}^n\left(\frac{1}{p_1}\right)$, with $\psi(\partial\mathcal{M}) \subset \partial\mathcal{B}^n\left(\frac{1}{p_1}\right)$ minimally and orthogonally (It is noteworthy that we do not require to presume that the associated mean curvature vector $\mathcal{H}_1 = \mathcal{H}$ of $\partial\mathcal{M}$ doesn't disintegrate systematically).

(ii) when $n = m$, assume that $\mathcal{M}$ is a bounded region of $\mathbb{R}^m$. Under these assumptions, for all admissible $r \in \{0, 1, \dots, m-1\}$, then

$$p_1 \left(\int_{\partial\mathcal{M}} \mathcal{H}_{r-1} dv_g \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} \mathcal{H}_r^2 dv_g \right) \quad (7)$$

Furthermore, for every admissible index $r \in \{0, 1, \dots, m-1\}$, equality is satisfied in Eq. (7) precisely when $\mathcal{M}$ is a Euclidean ball.

Comparable estimates are likewise derived for boundary compact spherical submanifolds. The issue regarding immersed manifolds of a negatively curved ambient space, a case that exhibits increased complexity, is yet unresolved (see [17]).

Assume that an n -manifold endowed with a Riemannian metric $(\mathcal{N}, \hbar)$ with sectional curvature equal to a constant is submerged isometrically by ψ in an orientable m -dimensional smooth manifold equipped with a Riemannian metric $(\mathcal{M}, g)$. Before defining several significant extrinsic geometric quantities, we go over the notion of generalized Kronecker delta symbols. Whenever $i_1, \dots, i_r$ as well as $j_1, \dots, j_r$ are numbers taking values in $\{1, \dots, m\}$, one has $\delta_{j_1, \dots, j_r}^{i_1, \dots, i_r}$ is either +1 or -1 depending on whether the indices $i_1, \dots, i_r$ are pairwise distinct and the indices $j_1, \dots, j_r$ make either of even or odd parity variation of the of $(i_1, \dots, i_r)$; otherwise, it equals zero. Such a matrix of $\mathcal{B}$ for a point $x \in \mathcal{M}$ with regard to an orthonormal framework $(e_i)_{1 \leq i \leq m}$ at the point x is $(\mathcal{B}_{ij} = \mathcal{B}(e_i, e_j))_{i, j \leq m}$. Suppose that $\mathcal{B}$ is the classical second form associated with the immersion ψ , where $\mathcal{B}$ takes values in the normal bundle. The mean curvature of order r associated with the immersion ψ is defined as the following for each $r \in \{1, \dots, m\}$.

(i) When r is even, the r -th mean curvature is described at the point x by a function $\mathcal{H}_r$, which is given by,

$$\mathcal{H}_r = \binom{m}{r}^{-1} \frac{1}{r!} \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_r}} \delta_{j_1, \dots, j_r}^{i_1, \dots, i_r} \hbar(\mathcal{B}_{i_1 j_1}, \mathcal{B}_{i_2 j_2}) \dots \hbar(\mathcal{B}_{i_{r-1} j_{r-1}}, \mathcal{B}_{i_r j_r})$$

and

(ii) For odd values of r , the r -th mean curvature defines a normal vector field at the point x , which is given by,

$$\mathcal{H}_r = \binom{m}{r}^{-1} \frac{1}{r!} \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_r}} \delta_{j_1, \dots, j_r}^{i_1, \dots, i_r} \hbar(\mathcal{B}_{i_1 j_1}, \mathcal{B}_{i_2 j_2}) \dots \hbar(\mathcal{B}_{i_{r-2} j_{r-2}}, \mathcal{B}_{i_{r-1} j_{r-1}}) \mathcal{B}_{i_r j_r}$$

Conventionally, we establish $\mathcal{H}_0 = 1$ and $\mathcal{H}_{m+1} = 0$. It is also noted that the mean curvature vector in the classical sense is $\mathcal{H}_1 = \mathcal{H}$. When $\mathcal{M}$ is narrowed by a unit normal vector arena v and the codimension of the immersion ψ is 1, working the second essential form viewed as a real-valued bilinear form $\mathcal{B}$ (i.e. $\mathcal{B}(X, Y) = \hbar(\mathcal{B}(X, Y), v)$). Thus, the odd-

order the r -th mean curvature is allowed to be indicated by a scalar quantity (substitute the normal bundle-valued field $\mathcal{H}_r$ with the scalar $\mathcal{H}(\mathcal{H}_r, v)$). Additionally, by selecting an orthonormal framework at the point x which diagonalizes $\mathcal{B}$ (i.e. $\mathcal{B}_x(e_i, e_j) = \mu_i \delta_{ij}$), this yields the subsequent statement for each integer r satisfying $1 \leq r \leq m$,

$$\mathcal{H}_r = \binom{m}{r}^{-1} \sum_{i_1 < \dots < i_r} \mu_{i_1} \dots \mu_{i_r}$$

in which the immersion ψ at x 's principal curvatures are indicated by (μ_i) .

For notational convenience, in the case where $\mathcal{N} = \mathbb{R}^{m+1}$, the description of $\mathcal{H}_r$ in the case $r = -1$ is extended by putting $\mathcal{H}_{-1} = -\langle \psi, v \rangle$.

Consider $(e_i)_{1 \leq i \leq m}$ as an orthonormal framework at a given point $x \in \mathcal{M}$, with $(e_i^*)_{1 \leq i \leq m}$ the corresponding coframe, and $(\mathcal{B}_{ij})$ given by the matrix of $\mathcal{B}$ at x in relation in the basis (e_i) . For $r \in \{1, \dots, m\}$, we certain the $(0, 2)$ -tensors $\mathcal{T}_r$ as follows :

(i) If r is even, we set

$$\mathcal{T}_r = \frac{1}{r!} \sum_{\substack{i, i_1, \dots, i_r \\ j, j_1, \dots, j_r}} \delta_{j, j_1, \dots, j_r}^{i, i_1, \dots, i_r} \mathcal{H}(\mathcal{B}_{i_1 j_1}, \mathcal{B}_{i_2 j_2}) \dots \mathcal{H}(\mathcal{B}_{i_{r-1} j_{r-1}}, \mathcal{B}_{i_r j_r}) e_i^* \otimes e_i^*$$

And

(ii) If r is odd, then we have

$$\mathcal{T}_r = \frac{1}{r!} \sum_{\substack{i, i_1, \dots, i_r \\ j, j_1, \dots, j_r}} \delta_{j, j_1, \dots, j_r}^{i, i_1, \dots, i_r} \mathcal{H}(\mathcal{B}_{i_1 j_1}, \mathcal{B}_{i_2 j_2}) \dots \mathcal{H}(\mathcal{B}_{i_{r-2} j_{r-2}}, \mathcal{B}_{i_{r-1} j_{r-1}}) \mathcal{B}_{i_r j_r} \otimes e_i^* \otimes e_i^*$$

Traditionally, $\mathcal{T}_0 = \mathcal{g}$. The two formulations can be consolidated for mean curvatures of order r when $\mathcal{M}$ is a hypersurface, namely $n = m + 1$. The scalar value is diagonalized if e_i constitutes an opposite frame at x and v represents the unit normal section, as established induced by the orientation of $\mathcal{M}$. In the second basic form $\mathcal{B}$, the tensors denoted by $\mathcal{T}_r$

may be regarded as scalar $(0, 2)$ tensors fields (upon substituting $\mathcal{T}_r$ with $\mathcal{H}(\mathcal{T}_r(\cdot, \cdot), v)$ for odd values of r), and this identity is obtained at the point x .

$$\mathcal{T}_r = \sum_{\substack{i < \dots < i_r \\ i_j \neq i}} \mu_{i_1} \dots \mu_{i_r} e_i^* \otimes e_i^*.$$

Utilizing the Codazzi–Mainardi equation as well as invariance of the fact that $\mathcal{N}$ has constant sectional curvature, it follows directly that $\text{div}_{\mathcal{M}} \mathcal{T}_r = 0$, for each admissible even index r , whenever the codimension exceeds one. The characteristics of the tensors $\mathcal{T}_r$ and the mean curvatures of order r are summed up as stated in [19].

Lemma 2.2: Let $r \in (1, \dots, m)$ be any integer, then we have

$$\text{tr}(\mathcal{T}_r) = k(r) \mathcal{H}_r$$

Furthermore, whenever r is an even integer

$$\sum_{i,j} \mathcal{T}_r(e_i, e_j) \mathcal{B}(e_i, e_j) = k(r) \mathcal{H}_{r+1}$$

and whenever r is an odd integer

$$\sum_{i,j} \mathcal{H}(\mathcal{T}_r(e_i, e_j) \mathcal{B}(e_i, e_j)) = k(r) \mathcal{H}_{r+1}$$

where $k(r) = (m - r) \binom{m}{r}$

We employ the real valued r -th mean curvatures of odd order if the codimension is 1, as previously mentioned, and set $\mathcal{H}_{-1} = -\langle \psi, \nu \rangle$ if $\mathcal{N} = \mathbb{R}^{m+1}$. Now let's review the the Minkowski-type integral formulas (generalization for arbitrary codimension is attributed to [2], [17], and the case of codimension 1 is attributed to [15]).

Lemma 2.3: (Minkowski-type integral formulas) When $\mathcal{M}$ has no boundaries and is compact, then we get

(i) In the case where $n > m + 1$ and $r \in \{1, 3, \dots, m\}$, we obtain

$$\int_{\mathcal{M}} (\mathcal{H}(\psi, \mathcal{H}_r) + \mathcal{H}_{r-1}) dv_g = 0.$$

(ii) In the case $n = m + 1$ and $r \in \{0, 1, 2, \dots, m\}$, then

$$\int_{\mathcal{M}} (\mathcal{H}(\psi, \mathcal{H}_r \nu) + \mathcal{H}_{r-1}) dv_g = 0.$$

3. Euclidean Reilly inequality

We shall now analyze the Reilly inequality in the Euclidean context. Initially, we suppose that $\mathcal{N} = \mathbb{R}^n$ possesses endowed with its canonical metric "can". We investigate the Steklov eigenvalue issue on a Riemannian manifold $\mathcal{M}$ that is compact and embedded within the Euclidean ambient space $\mathbb{R}^n$.

$$\begin{cases} \nabla f = 0 & \text{in } \mathcal{M} \\ \frac{\partial f}{\partial \nu} = \rho f & \text{on } \partial \mathcal{M} \end{cases} \quad (8)$$

The initial eigenvalue is zero, linked to constant as eigenfunctions, whereas that subsequent eigenvalue ρ_1 is defined variationally as follows

$$\rho_1 = \min_{\{f \in \mathcal{C}^1(\bar{\mathcal{M}}), \int_{\partial \mathcal{M}} f dv_g = 0\}} \frac{\int_{\mathcal{M}} |\nabla f|^2 dv_g}{\int_{\partial \mathcal{M}} f^2 dv_g} \quad (9)$$

where dv_g is volume component induced by the metric g on $\mathcal{M}$ and $\partial \mathcal{M}$. Furthermore, if

f is a $\mathcal{C}^1$ -function on $\bar{\mathcal{M}}$ satisfying $\int_{\partial \mathcal{M}} f dv_g = 0$, it follows that

$$p_1 = \frac{\int_{\mathcal{M}} |\nabla f|^2 dv_g}{\int_{\partial \mathcal{M}} f^2 dv_g}$$

if and only if the function f has the eigenvalue p_1 and is an eigenfunction associated with the Steklov spectrum Eq. (4).

In the sequel, $\mathcal{B}^n(r)$ will be represented as the n -dimensional Euclidean ball with radius r .

Theorem 3.1: Consider a compact Riemannian manifold endowed with a boundary, denoted as $(\mathcal{M}^m, g)$, which is isometrically immersed by ψ within $\mathbb{R}^n$, can). Let $\mathcal{H}_r$ represent the mean curvature of order r of the boundary $\partial \mathcal{M}$ of $\mathcal{M}$ in $\mathbb{R}^n$. We have

(i) Suppose $n > m$ and $r \in \{1, 3, \dots, m-1\}$, then

$$p_1 \left(\int_{\partial \mathcal{M}} \mathcal{H}_{r-1} dv_g \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial \mathcal{M}} |\mathcal{H}_r|^2 dv_g \right) \quad (10).$$

If additionally $\mathcal{H}_r$ doesn't vanish seamlessly, then equality in Eq. (10) for this choice of r holds if and only if ψ defines a minimal immersion of $\mathcal{M}$ into $\mathcal{B}^n\left(\frac{1}{p_1}\right)$, so that the boundary $\psi(\partial \mathcal{M})$ lies in $\partial \mathcal{B}^n\left(\frac{1}{p_1}\right)$ with orthogonal intersection, and the curvature vector $\mathcal{H}_r$ is collinear with ψ .

Specifically, in the case $r = 1$ in Eq. (10) occurs precisely when ψ constitutes realizes $\mathcal{M}$ as a minimal submanifold in $\mathcal{B}^n\left(\frac{1}{p_1}\right)$, with $\psi(\partial \mathcal{M}) \subset \partial \mathcal{B}^n\left(\frac{1}{p_1}\right)$ minimally and orthogonally (it is unnecessary to presume that the associated mean curvature vector $\mathcal{H}_1 = \mathcal{H}$ of $\partial \mathcal{M}$ doesn't be identically zero).

(ii) When $n = m$, one restrict consideration to the cases in which $\mathcal{M}$ constitutes a bounded region within $\mathbb{R}^m$. For each $r \in \{0, 1, \dots, m-1\}$, one has the following :

$$p_1 \left(\int_{\partial \mathcal{M}} \mathcal{H}_{r-1} dv_g \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial \mathcal{M}} \mathcal{H}_r^2 dv_g \right) \quad (11).$$

Additionally, equality is maintained in Eq. (11) for each $r \in \{0, 1, \dots, m-1\}$ if and only if $\mathcal{M}$ represents a ball.

Proof. By converting up to a translation of the immersion ψ , we may presume where

$\int_{\partial \mathcal{M}} \psi dv_g = 0$ (equivalently, for each $i = 1, 2, \dots, n$, $\int_{\partial \mathcal{M}} \psi_i dv_g = 0$). Summing on i and executing the variational specification Eq. (9) to the components ψ_i of ψ gives

$$p_1 \sum_{i=1}^n \left\{ \int_{\partial \mathcal{M}} \psi_i^2 dv_g \right\} \leq \sum_{i=1}^n \int_{\partial \mathcal{M}} |\nabla \psi_i|^2 dv_g = m(\text{Vol}(\mathcal{M}))$$

Or

$$p_1 \int_{\partial \mathcal{M}} |\phi|^2 dv_g \leq m(\text{Vol}(\mathcal{M})) \quad (12)$$

To demonstrate the Eq. (10) of the first declaration (i) inequality, Eq. (12) is multiplied by both sides via the value $\int_{\partial\mathcal{M}} |\mathcal{H}_r|^2 dv_g$, by invoking the Cauchy–Schwarz inequality is utilized to have

$$\begin{aligned} p_1 \left(\int_{\partial\mathcal{M}} \langle \phi, \mathcal{H}_r \rangle dv_g \right)^2 &\leq p_1 \left(\int_{\partial\mathcal{M}} |\phi| |\mathcal{H}_r| dv_g \right)^2 \\ &\leq p_1 \left(\int_{\partial\mathcal{M}} |\phi|^2 dv_g \right) \left(\int_{\partial\mathcal{M}} \mathcal{H}_r^2 dv_g \right) \\ &\leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_r|^2 dv_g \right) \end{aligned} \quad (13).$$

By employing the Minkowski-type integral formula (Lemma (2.3)) on $\partial\mathcal{M}$, regarded being compact submanifold of $\mathbb{R}^n$ (without boundary), we get the requisite inequality.

If $\mathcal{H}_r = 0$ (which cannot occur when $r = 1$), the Minkowski-type integral formula yields $\int_{\partial\mathcal{M}} \mathcal{H}_{r-1} dv_g = 0$. Equality is trivially satisfied in inequality Eq. (10).

Assume that $\mathcal{H}_r$ does not identically equal zero, which is invariably true when $r = 1$. If equality is satisfied in Eq. (10), then all the inequalities in Eq. (13) are, in reality, equalities. This suggests that for some constant $\lambda \neq 0$, we have $\mathcal{H}_r = \lambda\psi$ on $\partial\mathcal{M}$, and that all of the ψ_i components of ψ are eigenfunctions corresponding to p_1 . Given that $\mathcal{H}_r$ takes values in the normal bundle, ψ necessarily likewise lie in the normal bundle. Consequently, $X(|\psi|^2) = 2 \langle \nabla_X \psi, \psi \rangle = 0$ for all vector fields X tangent to $\partial\mathcal{M}$. It follows that both $|\psi|$ and $|\mathcal{H}_r|$ are constant on each connected component of $\partial\mathcal{M}$. Consequently, ψ sends linked components of the boundary $\partial\mathcal{M}$ onto hyperspherical subsets of $\mathbb{R}^n$ (observe that the associated r - t - $\hbar$ mean curvature vector $\mathcal{H}_r = \lambda\psi$ is indeed parallel normal-bundle valued, as $\nabla_X \psi$ is tangent to $\mathcal{M}$). Nonetheless, the immersion ψ is indeed minimal, and $v = p_1\psi$ on $\partial\mathcal{M}$, as the components ψ_i of ψ are eigenfunctions corresponding to p_1 . Consequently, the connecting boundary components $\partial\mathcal{M}$ intersect a common hypersphere in Euclidean space, characterized with a radius equal to $\frac{1}{p_1}$, at right angles.

Consider $f = \sum_{i=1}^n \psi_i^2$ now. Given that $\mathcal{M}$ a minimum submanifold of $\mathbb{R}^n$, a straightforward computation yields the Laplacian of f satisfies $\Delta f = 2m$. Therefore, it follows that

$$\begin{cases} \Delta f = 2m & \text{in } \mathcal{M} \\ f = \left(\frac{1}{p_1}\right)^2 & \text{on } \partial\mathcal{M} \end{cases} \quad (14)$$

Consequently, by employing the maximal principle, we infer that

$$f \leq \left(\frac{1}{p_1}\right)^2 \text{ on } \mathcal{M}, \text{ indicating}$$

$$\psi(\mathcal{M}) \subset \mathcal{B}^n\left(\frac{1}{p_1}\right).$$

Conversely, Each coordinate function ψ_i of the immersion ψ serves as a Steklov eigenfunction associated with the eigenvalue p_1 , since $\mathcal{M}$ is a minimum submanifold of

$\mathbb{R}^n$ and its boundary $\partial\mathcal{M}$ intersects the hypersphere $\partial\mathcal{B}^n\left(\frac{1}{p_1}\right)$ orthogonally. Applying the proportionality hypothesis between $\mathcal{H}_r$ and ψ also results in equality for all of the inequalities shown in (13). According to the Takahashi theorem (see [4]), we can conclude that for $r = 1$, Because ψ and $\mathcal{H}$ are proportionate, ψ realizes a minimum immersion into a Euclidean hypersphere. This completes the proof of assertion (i) is therefore finished.

The proof of Theorem (3.1)'s second assertion (ii) is similar to that of the first assertion, with the exception of the equality case associated with $r = 0$. In the specific case where $r = 0$, it is sufficient to employ the following in place of inequality Eq. (13) :

$$\begin{aligned} p_1 \left(\int_{\partial\mathcal{M}} \langle \psi, v \rangle dv_g \right)^2 &\leq p_1 \left(\int_{\partial\mathcal{M}} |\psi| |v| dv_g \right)^2 \leq p_1 \left(\int_{\partial\mathcal{M}} |\psi|^2 dv_g \right) \left(\int_{\partial\mathcal{M}} |v|^2 dv_g \right) \\ &\leq m \text{Vol}(\mathcal{M}) \text{Vol}(\partial\mathcal{M}) \\ &= m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_0|^2 dv_g \right) \end{aligned} \quad (15).$$

So, for some $r \in \{0, \dots, m-1\}$, if the equality is satisfied, then $v = p_1\psi$, and consequently, the boundary $\partial\mathcal{M}$ coincides with a hypersphere in $\mathbb{R}^m$. To conclude, $\mathcal{M}$ constitutes a ball in Euclidean space (recall that for the Euclidean ball $\mathcal{B}^m(r)$, of radius r one has $p_1 = \frac{1}{r}$). Conversely, to confirm the equality case of Eq. (11) for the Euclidean ball, it is sufficient, due to homogeneity, to validate it for the unit ball. It is evident in the case $r = 0$, as well as for all integers r with $1 \leq r \leq m-1$, the claim is derived which follows from the properties of the unit $(m-1)$ -sphere $\mathbb{S}^{m-1}(1)$, where $\mathcal{H}_r = 1$, and the Euclidean unit ball $\mathcal{B}^m(1)$, where $p_1 = 1$. Furthermore, $\text{Vol}(\mathbb{S}^{m-1}(1)) = m \text{Vol}(\mathcal{B}^m(1))$.

Theorem (3.1) gives rise to two relevant outcomes. Among them, the first concerns a classical isoperimetric upper bound for the Steklov eigenvalue p_1

Corollary 3.2: Take $\mathcal{M}$ as a bounded domain of $\mathbb{R}^m$. Leads to

$$p_1 \leq \frac{\text{Vol}(\partial\mathcal{M})}{m \text{Vol}(\mathcal{M})} \quad (16)$$

Furthermore, equality occurs precisely in the case where $\mathcal{M}$ is a ball.

Proof. It is sufficient to invoke the second part of Theorem (3.1) for the case $r = 0$, noting that $\left| \int_{\partial\mathcal{M}} \mathcal{H}_{-1} dv_g \right| = m \text{Vol}(\mathcal{M})$, which follows directly from an application of the Gauss divergence theorem to the vector field $X = \psi$ on $\mathcal{M}$.

It is important to note that the Weinstock inequality Eq. (5) is preferable to inequality Eq. (16) in two dimensions and when the domain possesses a linked boundary. A direct consequence of the isoperimetric inequality produces

$$\frac{\text{Vol}(\partial\mathcal{M})^2}{\text{Vol}(\mathcal{M})} \geq 4\pi,$$

Consequently, by employing the Weinstock inequality, we obtain

$$p_1 \leq \frac{2\pi}{\text{Vol}(\partial\mathcal{M})} = \frac{4\pi}{2\text{Vol}(\partial\mathcal{M})} \leq \frac{\text{Vol}(\partial\mathcal{M})}{2\text{Vol}(\mathcal{M})}$$

which corresponds to inequality Eq. (16).

In the case $r = 1$, Theorem 3.1 reduces to a Reilly inequality expressed via the mean curvature, which forms the content of the second corollary.

Corollary 3.3: Let $\mathcal{M}$ be a compact Riemannian manifold with boundary, and let $\psi: (\mathcal{M}, g) \rightarrow (\mathbb{R}^n, can)$ be an isometric immersion. Denote by $\mathcal{H}$ the mean curvature of the boundary $\partial\mathcal{M}$ viewed as a hypersurface in $\mathbb{R}^n$. Then one obtains

$$p_1 \leq m \frac{\text{Vol}(\mathcal{M})}{2\text{Vol}(\partial\mathcal{M})^2} \int_{\partial\mathcal{M}} |\mathcal{H}|^2 dv_g \quad (17)$$

Moreover,

(i) For $n > m$, equality in Eq. (17) is attained precisely when ψ realizes $\mathcal{M}$ as a minimal immersion of $\mathcal{B}^n\left(\frac{1}{p_1}\right)$, with $\psi(\partial\mathcal{M}) \subset \partial\mathcal{B}^n\left(\frac{1}{p_1}\right)$ meeting the boundary orthogonally.

(ii) In the case $n = m$ and when $\mathcal{M}$ is a region in $\mathbb{R}^m$, equality in Eq. (17) is attained precisely when $\mathcal{M}$ is a ball.

Observe that :

- Inequality Eq. (17) naturally yields a corresponding inequality for submanifolds in spheres and projective spaces. Utilizing the conventional ways that these spaces are embedded in a Euclidean framework (see to [10]), a straightforward calculation produces the following results:

For an m -dimensional compact Riemannian submanifold $(\mathcal{M}, g)$ with boundary, isometrically immersed in $(\mathcal{N}, h)$, the Steklov eigenvalue p_1 obeys the following inequality. This submanifold is equipped with its corresponding canonical metric and represents the sphere $\mathbb{S}^n$, the real projective space $\mathbb{R}\mathcal{P}^n$, the complex projective space $\mathbb{C}\mathcal{P}^n$, or the quaternionic projective space $\mathbb{Q}\mathcal{P}^n$

$$p_1 \leq m \frac{\text{Vol}(\mathcal{M})}{\text{Vol}(\partial\mathcal{M})^2} \int_{\partial\mathcal{M}} (|\mathcal{H}|^2 + c(m)dv_g) \quad (18)$$

Here, $\mathcal{H}$ denotes the mean curvature vector of the boundary $\partial\mathcal{M}$ regarded as a submanifold of $\mathcal{N}$, and $c(m)$ is defined by

$$c(m) = \begin{cases} 1, & \text{if } \mathcal{N} = \mathbb{S}^n \\ \frac{2(m+1)}{m}, & \text{if } \mathcal{N} = \mathbb{R}\mathcal{P}^n \\ \frac{2(m+2)}{m}, & \text{if } \mathcal{N} = \mathbb{C}\mathcal{P}^n \\ \frac{2(m+4)}{m}, & \text{if } \mathcal{N} = \mathbb{Q}\mathcal{P}^n \end{cases}$$

- Adapting the inequalities related to the mean curvatures of order $r \geq 2$ is not straightforward, even for spherical submanifolds.
- To express inequality Eq. (17), may be reformulated by expressing via the mean curvature $\mathcal{H}^{\partial\mathcal{M}}$ of $\partial\mathcal{M}$ in $\mathcal{M}$ and the mean curvature $\mathcal{H}^{\mathcal{M}}$ of $\mathcal{M}$ in $\mathbb{R}^n$. In this formulation, the mean curvature $\mathcal{H}$ of $\partial\mathcal{M}$ in $\mathbb{R}^n$ is required to satisfy

$$|\mathcal{H}|^2 = |\mathcal{H}^{\mathcal{M}}|^2 + |\mathcal{H}^{\partial\mathcal{M}}|^2.$$

- Scalar curvature $Scal_{\partial\mathcal{M}}$ of the boundary $\partial\mathcal{M}$ can be bounded via inequality Eq. (11) in the case $r = 2$ with $m \geq 3$. Indeed, an application of the Gauss equation yields $Scal_{\partial\mathcal{M}} = (m-1)(m-2)\mathcal{H}_2$. Eq. (11) for inequality thus turns into

$$p_1 \left(\int_{\partial\mathcal{M}} \mathcal{H} dv_g \right)^2 \leq \frac{m}{((m-1)(m-2))^2} \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} Scal_{\partial\mathcal{M}}^2 dv_g \right)$$

In spherical case, despite some challenges, we will demonstrate how we are able to successfully extend our Theorem (3.1) established for Euclidean submanifolds to the spherical ones.

First, one will present some concepts and indications. Suppose that $(\mathcal{M}, g)$ is a Riemannian manifold of dimension m . (denoted as $\partial\mathcal{M}$ hereafter) that permits an isometric immersion in $(\mathbb{R}^n, can)$, and designate its second fundamental form by $\mathcal{B}$, as previously indicated. To $(\mathcal{M}, g)$, we give a symmetric tensor field $\mathcal{T}$ of type (0,2) that is divergence-free and set up a normal vector field $\mathcal{H}_{\mathcal{T}}$ defining it as follows:

$$\mathcal{H}_{\mathcal{T}}(x) = \sum_{1 \leq i, j \leq m} \mathcal{T}(e_i, e_j) \mathcal{B}(e_i, e_j)$$

in which $(e_i)_{1 \leq i \leq m}$ forms an orthonormal frame of $\mathcal{T}_x\mathcal{M}$. We now introduce the second-order differential operator $\mathcal{L}_{\mathcal{T}}$ which works on $\mathcal{C}^\infty(\mathcal{M})$ as follows, will now be presented:

$$\mathcal{L}_{\mathcal{T}}(f) = -\text{div}_{\mathcal{M}}(\mathcal{T}^\# \nabla^{\mathcal{M}} f)$$

In this context, $\mathcal{T}^\#$ is the (1,1)-tensor canonically associated with $\mathcal{T}$ through the metric g , while $\nabla^{\mathcal{M}}$ represents the gradient operator on $(\mathcal{M}, g)$. Moreover, $(\epsilon_i)_{1 \leq i \leq n}$ denotes the canonical basis of $\mathbb{R}^n$, and $(\psi_i)_{i \leq n}$ are the component functions of the immersion ψ . One defines $\mathcal{L}_{\mathcal{T}}\psi$ by $\mathcal{L}_{\mathcal{T}}\psi = \sum_{i=1}^n (\mathcal{L}_{\mathcal{T}}\psi_i) \epsilon_i$. A direct calculation (refer to [5]) yields $\mathcal{L}_{\mathcal{T}}\psi = -\mathcal{H}_{\mathcal{T}}$ and $\frac{1}{2} \mathcal{L}_{\mathcal{T}}|\psi|^2 = -\langle \psi, \mathcal{H}_{\mathcal{T}} \rangle - \text{tr}(\mathcal{T})$. For a closed submanifold $\mathcal{M}$ of $\mathbb{R}^n$, the Hsiung-Minkowski formulas are generalized by this final identity (see [5], [19]).

$$0 = \frac{1}{2} \int_{\partial\mathcal{M}} \mathcal{L}_{\mathcal{T}}|\psi|^2 dv_g = - \int_{\mathcal{M}} \langle \psi, \mathcal{H}_{\mathcal{T}} \rangle dv_g - \int_{\mathcal{M}} \text{tr}(\mathcal{T}) dv_g \quad (19)$$

Indeed, this formula can be used to derive the Minkowski-type integral formula outlined in Lemma (2.3) by applying the features of $\mathcal{T}_r$ given in Lemma (2.2) and then substituting $\mathcal{T} = \mathcal{T}_r$. By substituting Eq. (19) into $\partial\mathcal{M}$ in lieu of the Minkowski-type integral formula (Lemma (2.3)), The following result constitutes a generalization of Theorem (3.1).

Theorem 3.4: Assume that $\psi : (\mathcal{M}^m, g) \rightarrow (\mathbb{R}^n, can)$ denote an isometric immersion of a compact Riemannian manifold $(\mathcal{M}, g)$ possessing a boundary. Let $\mathcal{T}$ be a symmetric divergence-free (0,2)-tensor defined on $\partial\mathcal{M}$, and denote by $\mathcal{H}_{\mathcal{T}}$ the associated normal vector field. Then one has:

$$p_1 \left(\int_{\partial\mathcal{M}} (\text{tr}\mathcal{T}) dv_g \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_r|^2 dv_g \right) \quad (20)$$

Furthermore, if $\mathcal{H}_{\mathcal{T}}$ does not equate to zero universally, then we have

(i) Let $n > m$, so equality in Eq. (20) is satisfied if and only if ψ immerses $\mathcal{M}$ to a minimal extent in $\mathcal{B}^n\left(\frac{1}{p_1}\right)$, in such a way that $\psi(\partial\mathcal{M}) \subset \partial\mathcal{B}^n\left(\frac{1}{p_1}\right)$ intersects perpendicularly and $\mathcal{H}_{\mathcal{T}}$ is parallel to ψ .

(ii) In the case $n = m$, when $\mathcal{M}$ is a bounded region in $\mathbb{R}^n$, equality holds precisely if trace $\text{tr}(\mathcal{T})$ is the constant and $\mathcal{M}$ is a ball.

Proof. Eq. (20) follows by multiplying both sides of Eq. (12) followed by the amount $\int_{\partial\mathcal{M}} |\mathcal{H}_{\mathcal{T}}|^2$, and subsequently applying the Cauchy–Schwarz inequality, which yields

$$\begin{aligned} p_1 \left(\int_{\partial\mathcal{M}} \langle \psi, \mathcal{H}_{\mathcal{T}} \rangle dv_g \right)^2 &\leq p_1 \left(\int_{\partial\mathcal{M}} |\psi| |\mathcal{H}_{\mathcal{T}}| dv_g \right)^2 \\ &\leq p_1 \left(\int_{\partial\mathcal{M}} |\psi|^2 dv_g \right) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_{\mathcal{T}}|^2 dv_g \right) \\ &\leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_{\mathcal{T}}|^2 dv_g \right) \end{aligned} \quad (21).$$

but by Eq. (19), we have

$$\int_{\partial\mathcal{M}} \langle \psi, \mathcal{H}_{\mathcal{T}} \rangle dv_g = - \int_{\partial\mathcal{M}} \text{tr}(\mathcal{T}) dv_g$$

By presenting this ultimate equality in Eq. (21), we achieve the requisite inequality.

As follows from the argument of Theorem (3.1), when $\mathcal{H}_{\mathcal{T}} = 0$, equality in Eq. (20) is trivially fulfilled. Assume that $\mathcal{H}_{\mathcal{T}}$ is not identically zero throughout. the equality case of Eq. (20) occurs if and only if $\mathcal{H}_{\mathcal{T}}$ is a scalar multiple of ψ , and the coordinate functions ψ_i of ψ being eigenfunctions associated with the eigenvalue p_1 . When $n > m$, we can deduce a conclusion as illustrated in Theorem (3.1). When $\mathcal{M}$ in $\mathbb{R}^m$ is a bounded region, The state $v = p_1\psi$ indicates that $\partial\mathcal{M}$ lies on a hypersphere with a radius of $\frac{1}{p_1}$, so $\mathcal{M} = \mathcal{B}^m\left(\frac{1}{p_1}\right)$. By contrast, $\mathcal{H}_{\mathcal{T}} = \text{tr}(\mathcal{T})v$, where $\text{tr}(\mathcal{T})$ constant. In return, when $\text{tr}(\mathcal{T})$ is constant, the equality case of Eq. (20) for the ball of Euclidean units is sufficient to be verified. With $\text{Vol}(\mathbb{S}^{m-1}(1)) = m \text{Vol}(\mathcal{B}^m(1))$, we can derive this because the unit hypersphere $\mathbb{S}^{m-1}(1)$ is completely umbilicity.

Theorem 3.5: Suppose a compact Riemannian manifold with boundary, denoted by $(\mathcal{M}^m, g)$, where ψ is isometrically submerged in $(\mathbb{S}^n, \text{can})$. For the boundary $\partial\mathcal{M}$ in $\mathbb{S}^n$, let $\mathcal{H}_r$ be the r -th mean curvatures; we get

(i) If $n > m$ and $r \in \{1, 3, \dots, m-1\}$, then

$$p_1 \left(\int_{\partial\mathcal{M}} \mathcal{H}_{r-1} dv_g \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} (|\mathcal{H}_{r-1}|^2 + |\mathcal{H}_r|^2) dv_g \right) \quad (22)$$

(ii) Assuming $n = m$, we may restrict our consideration into the scenario $\mathcal{M}$ is a bounded region of $\mathbb{S}^n$. If r is an element of the set $\{1, 2, \dots, m-1\}$, then

$$p_1 \left(\int_{\partial\mathcal{M}} \mathcal{H}_{r-1} dv_g \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} (\mathcal{H}_{r-1}^2 + \mathcal{H}_r^2) dv_g \right) \quad (23)$$

Furthermore, unless $\mathcal{H}_r = \mathcal{H}_{r-1} = 0$, equality in Eq. (22) or Eq. (23) is impossible.

Proof. The second fundamental form of ψ is denoted by $\mathcal{B}$. Let $\mathcal{B}'$ represent the second basic form associated with $i \circ \psi$. Assume that i denotes the canonical embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$. As previously stated, we show the normal vector domain $\mathcal{H}'_{\mathcal{T}_r}$ related associated with $\mathcal{B}'$, which at $x \in \partial\mathcal{M}$ is defined by

$$\mathcal{H}'_{\mathcal{T}_r} = \sum_{1 \leq i, j \leq m} \mathcal{T}_r(e_i, e_j) \mathcal{B}'(e_i, e_j)$$

in which the orthonormal frame at x is given by $(e_i)_{1 \leq i \leq m}$. Consequently, by applying Theorem (3.5) with $i \circ \psi$, the we get

$$p_1 \left(\int_{\partial\mathcal{M}} \mathcal{T}_r(\mathcal{T}_{r-1}) dv_{\mathcal{G}} \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} |\mathcal{H}'_{\mathcal{T}_{r-1}}|^2 dv_{\mathcal{G}} \right) \quad (24)$$

Since $\mathcal{G} \otimes \psi = \mathcal{B} - \mathcal{B}'$ it follows that $\mathcal{T}_r(\mathcal{T}_{r-1})\psi = \mathcal{H}_{\mathcal{T}_{r-1}} - \mathcal{H}'_{\mathcal{T}_{r-1}}$. Therefore,

$$\mathcal{T}_r(\mathcal{T}_{r-1})^2 = |\mathcal{H}_{\mathcal{T}_{r-1}}|^2 - |\mathcal{H}'_{\mathcal{T}_{r-1}}|^2$$

By reporting the final equality in Eq. (24), we obtain

$$p_1 \left(\int_{\partial\mathcal{M}} \mathcal{T}_r(\mathcal{T}_{r-1}) dv_{\mathcal{G}} \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_{\mathcal{T}_{r-1}}|^2 + \mathcal{T}_r(\mathcal{T}_{r-1})^2 dv_{\mathcal{G}} \right) \quad (25)$$

Lastly, inequality Eq. (22) is obtained by applying Lemma (2.2), that offers $\mathcal{T}_r(\mathcal{T}_{r-1}) =$

$$k(r-1)\mathcal{H}_{r-1} \text{ and } |\mathcal{H}_{\mathcal{T}_{r-1}}| = k(r-1)|\mathcal{H}_r|.$$

Assuming that $\mathcal{H}_r \neq 0$ or $\mathcal{H}_{r-1} \neq 0$ (i.e., $\mathcal{H}'_{\mathcal{T}_{r-1}} \neq 0$), we can deduce from the preceding proof that $i \circ \psi$ immerses $\mathcal{M}$ minimally in $\mathbb{R}^{n+1}$ if equality holds. This cannot occur; indeed, denoting designate by $\mathcal{H}'_{\mathcal{M}}$ and $\mathcal{H}_{\mathcal{M}}$ the mean curvatures associated with $i \circ \psi$ and ψ , respectively, it follows that

$$|\mathcal{H}_{\mathcal{M}}|^2 + 1 = 0 = |\mathcal{H}'_{\mathcal{M}}|^2.$$

Remark 3.6: Theorem (3.5) provides disparity when $r = 1$.

$$p_1 \leq m \frac{\text{Vol}(\mathcal{M})}{\text{Vol}(\partial\mathcal{M})^2} \int_{\partial\mathcal{M}} (|\mathcal{H}|^2 + 1) dv_{\mathcal{G}} \quad (26)$$

which was already observed in Note that above.

Corollary 3.7: Consider an orthogonal immersion $\psi_j : (\mathcal{M}^m, \mathcal{G}) \rightarrow (\mathbb{R}^n, \text{can})$ of a compact Riemannian manifold with boundary. Suppose that $\mathcal{T}$ is a divergence-free symmetric tensor of type (0,2) on $\partial\mathcal{M}$, and let $\mathcal{H}_{\mathcal{T}}$ denote the corresponding vector field. One then has

$$p_1 \left(\int_{\partial\mathcal{M}} (\mathcal{T}_r \mathcal{T}) dv_{\mathcal{G}_j} \right)^2 \leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_r|^2 dv_{\mathcal{G}_j} \right) \quad (27)$$

Furthermore, if $\mathcal{H}_{\mathcal{T}}$ doesn't precisely equal zero, then we derive

(i) For $n > m$, equality in Eq. (27) is attained precisely when each map ψ_j realizes a minimal immersion of $\mathcal{M}$ into $\mathcal{B}_j^n\left(\frac{1}{p_1}\right)$, such that

$$\sum_{j=1}^n \psi_j(\partial\mathcal{M}) \subset \sum_{j=1}^n \mathcal{B}_j^n\left(\frac{1}{p_1}\right)$$

with orthogonal intersection, and the vector field $\mathcal{H}_{\mathcal{T}}$ is a scalar multiple of ψ_j .

(ii) In the case $n = m$, when $\mathcal{M}$ is a bounded region of $\mathbb{R}^n$, the equality condition is met precisely when $\text{tr}(\mathcal{T})$ is constant and $\mathcal{M}$ coincides with a ball.

Proof. By multiplying Eq. (12) on both sides by the integral $\int_{\partial\mathcal{M}} |\mathcal{H}_{\mathcal{T}}|^2$ and then invoking the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned} p_1 \left(\int_{\partial\mathcal{M}} \sum_{j=1}^n \langle \psi_j, \mathcal{H}_{\mathcal{T}} \rangle dv_{g_j} \right)^2 &\leq p_1 \left(\int_{\partial\mathcal{M}} \sum_{j=1}^n |\psi_j| |\mathcal{H}_{\mathcal{T}}| dv_{g_j} \right)^2 \\ &\leq p_1 \left(\int_{\partial\mathcal{M}} \sum_{j=1}^n |\psi_j|^2 dv_{g_j} \right) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_{\mathcal{T}}|^2 dv_{g_j} \right) \\ &\leq m \text{Vol}(\mathcal{M}) \left(\int_{\partial\mathcal{M}} |\mathcal{H}_{\mathcal{T}}|^2 dv_{g_j} \right) \end{aligned} \quad (28).$$

but by Eq. (19), we have

$$\int_{\partial\mathcal{M}} \sum_{j=1}^n \langle \psi_j, \mathcal{H}_{\mathcal{T}} \rangle dv_{g_j} = - \int_{\partial\mathcal{M}} \text{tr}(\mathcal{T}) dv_{g_j}$$

we obtain the desired results. Moreover we can see proof of Theorem (3.1), for the rest of the proof

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