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FULL PAPER

Ion Acoustic Solitary Wave Solutions in the Context of the Nonlinear Fractional KdV Equation

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Abstract:

This manuscript provides a comprehensive investigation of the fractional (1+1)-dimensional nonlinear Korteweg-de Vries (KdV) model in the context of quantum plasma. The primary objective is to derive and mathematically explain the fractional (1+1)-dimensional nonlinear KdV model within quantum plasmas, focusing on ion acoustic solitary waves. Using the reductive perturbation technique, the fractional KdV equation is formulated and solved via the tanh hyperbolic function technique. The study examines the effects of ion pressure and an external electric field on ion acoustic waves, considering plasma characteristics such as ion-electron temperature ratios, nonextensive electron and positron effects. The manuscript investigates the influence of the fractional order and plasma parameters on the phase velocity of ion acoustic waves. Notably, the results reduce to known outcomes when the fractional order equals one. This work contributes significantly to the understanding of nonlinear phenomena in quantum plasmas, particularly ion acoustic waves, with potential applications in astrophysical and cosmological contexts.

Keywords: KdV model, fractional calculus, quantum plasma, solitary waves, ion acoustic

1. Introduction

Quantum plasmas, which involve the collective behavior of charged particles under the influence of quantum mechanical effects, have become an important subject of study in various fields, including astrophysics, space physics, and fusion research. To describe the behavior of quantum plasmas, three well-established models are commonly used, each incorporating different physical effects and mathematical frameworks [G. C. Das and S. N. Paul, 1985], [W. Masood, N. Jehan, A. M. Mirza, and P. H. Sakanaka, 2008]. The quantum hydrodynamics (QHD) model is one of the most widely used and fundamental frameworks in the study of quantum plasmas. It provides a macroscopic description by combining the classical fluid equations with quantum mechanical principles such as quantum pressure, quantum potential, and quantum statistics. The QHD model serves as a basis for understanding the collective behavior of quantum plasma systems, including phenomena such as wave propagation, particle dynamics, and plasma instabilities. In this model, the plasma's collective behavior is described by a set of fluid-like equations, including the continuity equation, Euler equation, and Poisson equation, with quantum corrections that account for the wave nature of particles and the effects of quantum statistics [T. S. Gill, A. Singh, H. Kaur, N. S. Saini, and P. Bala, 2007], [M. G. Hafez and M. R. Talukder, 2015]. In contrast, the classical plasma fluid model extends the principles of classical fluid dynamics to describe plasma systems. While it does not incorporate quantum effects, it remains essential for understanding the macroscopic behavior of classical plasmas. The classical equations of motion, such as the continuity equation and the Navier-Stokes equation govern the plasma's evolution in the absence of quantum effects and can be viewed as a limiting case of the QHD model when quantum effects are negligible. The classical model forms the foundation for many plasma theories and is still widely applied to describe low-temperature plasmas where quantum effects are not dominant [M. G. Hafez, M. R. Talukder, and R. Sakthivel, 2016] ,[H. K. Malik, 1996.].

An important area of research in quantum plasmas involves the formulation and analysis of nonlinear fractional differential equations (FDEs) or partial differential equations (PDEs), which are used to model the wave propagation and structural dynamics within quantum plasma systems. Nonlinearities and fractional calculus provide an effective framework for describing complex phenomena, such as solitons, shock waves, and wave-particle interactions. By combining the QHD model with these mathematical techniques, researchers can develop more accurate models that account for the nonlinear and fractional effects observed in quantum plasma behavior [H. K. Malik, 1995], [K. Singh, V. Kumar, and H. K. Malik, 2005.]. These models provide a comprehensive approach for understanding the behavior of quantum plasmas, ranging from microscopic quantum effects to macroscopic plasma dynamics. This integrated framework allows researchers to investigate the rich variety of phenomena in quantum plasmas, such as quantum shock waves, solitary waves, and other nonlinear plasma behaviors, with applications in fields such as space physics, fusion energy, and astrophysical systems [H. K. Malik, 1996] , [D. K. Singh and H. K.Malik,2005].

The study of magnetized ion-electron-positron quantum plasmas has become a focal point in understanding the behavior of ion-acoustic (IA) waves, both in linear and nonlinear regimes. These plasmas composed of electrons, positrons, and ions, exhibit complex wave dynamics that are influenced by quantum mechanical effects. A key area of investigation involves the analysis of ion-acoustic waves (IAWs), which are a type of sound wave propagating through the plasma, and their response to both classical and quantum effects.

Recent studies have expanded our understanding of IAWs by utilizing the QHD model, which incorporates quantum mechanical corrections into the classical fluid equations. Khan and Mushtaq explored the properties and stability of IAWs in ultracold quantum plasmas, including the effects of transverse perturbations on wave propagation. By deriving the KP equation, they were able to study the dynamics of IAWs under these conditions, providing insights into their stability and nonlinear evolution. Khan and Haque employed the QHD model to derive the nonlinear weakly limit of the deformed KdV Burgers equation, which is crucial for understanding the nonlinear dynamics of IAWs in quantum plasmas. This work highlights the importance of nonlinear modeling in describing the complex behavior of ion-acoustic waves, particularly in quantum environments [R. Malik, H. K. Malik, and S. C. Kaushik, 2012] , [F. C. Michel, 1982]. These investigations demonstrate the ongoing efforts to develop and refine mathematical models, such as the QHD model and its extensions, to better understand the behavior of ion-acoustic waves and related phenomena in quantum plasmas. The exploration of nonlinear equations and the study of wave dynamics continue to be pivotal in advancing our understanding of quantum plasmas and their applications in various fields, from fusion energy to astrophysical environments [G. C. Das and S. N. Paul, 1985] , [H. R. Pakzad and M. Tribeche, 2013].

The study of nonlinear waves, particularly solitons and solitary waves, is of paramount importance across various scientific disciplines, from laboratory research to astrophysical and space physics. These waves, which maintain their shape while propagating over time, are integral to understanding the dynamics of complex plasma systems. Nonlinear waves have been observed in a wide range of phenomena, including polar magnetospheres, solar wind, and the Earth's magnetotail, underscoring their relevance in both natural and experimental plasma environments. In these contexts, solitons and solitary waves provide valuable insights into the nonlinear and dynamic behavior of plasmas, which can exhibit both positive and negative wave amplitudes depending on the system parameters. Solitons and solitary waves are key solutions to nonlinear PDEs and are characterized by their stability and ability to maintain their shape while propagating over long distances. These waves often arise in spatially extended systems where nonlinearity dominates the wave propagation. Their significance lies not only in their theoretical properties but also in their potential practical applications, which span laboratory experiments and astrophysical observations. Understanding the formation, propagation, and interaction of these waves is critical for a wide range of applications, from controlling plasma in fusion reactors to understanding cosmic plasma phenomena.

Fractional calculus has emerged as a powerful and transformative tool in mathematics, offering more accurate and flexible models for real-world phenomena compared to traditional integer-order calculus. Its primary strength lies in its ability to effectively describe systems with non-local or long-range memory effects, which are often inadequately represented by classical models. The field has rapidly expanded, with multiple definitions of fractional derivatives developed to suit different applications, including the Riemann–Liouville, Caputo, and conformable fractional derivatives. Each of these definitions provides unique advantages in various domains such as physics, engineering, biology, and finance [A.A. Kilbas, H.M. Srivastava, J.J. Trujillo,2006] , [M G Hafez, M R Talukder, and M H Ali, 2016]. One significant advancement in the field is the introduction of the conformable fractional derivative (CFD) by Khalil et al. [E. A-B. Abdel-Salam, M.F. Mourad, Math.2018], which extends the idea of fractional differentiation by incorporating simpler computational techniques while preserving the key properties of fractional derivatives. As research in fractional calculus continues to evolve, it holds great promise for advancing mathematical modeling in the natural and applied sciences, making it a cornerstone of modern scientific analysis

Khalil et al. [E. A-B. Abdel-Salam, M.F. Mourad, Math.2018] introduced the CFD by establishing its definition through a limit process, offering a simpler and more computationally efficient approach compared to traditional fractional derivatives,

$$D^{\beta}\psi(x)=\lim_{\varepsilon\to 0}\frac{\psi(x+\varepsilon x^{1-\beta})-\psi(x)}{\varepsilon},\quad \forall x>0,\quad \beta\in(0,1].$$

Substituting $\alpha=1$ into the final equations, the noninteger differentials transition into the well-established integer differentials. Unlike classical definitions, which often involve integrals or complex limiting processes, the conformable fractional derivative is defined in such a way that it can be applied directly to functions in a more straightforward manner, while still capturing the key properties of fractional differentiation. This makes the CFD particularly useful for solving fractional differential equations in various fields, as it preserves the non-local memory effects characteristic of fractional models but with less computational complexity. The introduction of the CFD has opened up new possibilities for both theoretical research and practical applications, particularly in systems where classical fractional calculus may be challenging to apply.

This paper is organized as follows: Section 2 presents the derivation of the basic fractional KdV model. In Section 3, we outline the procedure for obtaining and constructing explicit solitary wave solutions for the model. Section 4 is dedicated to a discussion of the results. Finally, in the concluding section, we summarize the key findings and offer some prospects for future research.

2. The formulation of the derivation for the KdV model

We investigate the propagation of fractional nonlinear shocks and IASWs in a fully ionized, unmagnetized, three-component plasma system composed of relativistic hot ions, positrons, and nonextensive electrons. It is expected that the spatial fractional speed of quantum ions in acoustics will be significantly higher than any spatial fractional speed associated with the plasma flow. The charge neutrality equilibrium condition is assumed to be $n_{e0} = n_{i0} + n_{p0}$, where the concentrations of ions, positrons, and unperturbed electrons are denoted by n_{p0}, n_{i0} and n_{e0}, respectively. Additionally, it is assumed that the electron and positron concentrations follow an equilibrium q-distribution function. It is possible to acquire the normalized non extensive concentrations of positron and electron [Wang M. L. and Li X. Z., 2005] as

$$n_e = \frac{1}{1-a}[1+(q-1)\varphi]^{\frac{q+1}{2(q-1)}}$$
$$= \frac{1}{1-a}\left(1+\frac{1}{2}(q+1)\varphi - \frac{1}{8}(q+1)(q-3)\varphi^2 + \frac{1}{48}(q+1)(\mathrm{q}-3)(3\mathrm{q}-5)\varphi^3 + ...\right),$$

$$n_p = \frac{a}{1-a}[1-\sigma(q-1)\varphi]^{\frac{q+1}{2(q-1)}}$$
$$= \frac{a}{1-a}\left(1-\frac{\sigma}{2}(q+1)\varphi - \frac{\sigma^2}{8}(q+1)(q-3)\varphi^2 - \frac{\sigma^3}{48}(q+1)(\mathrm{q}-3)(3\mathrm{q}-5)\varphi^3 + ...\right),$$

Where $a = n_{p0}/n_{e0}$ and $\sigma = T_e/T_p$. In n_e and n_p, we use q tends to 1 for isothermal electrons and positrons, $q > 1$ for sub-thermal electrons and $-1 < q < 1$ for super-thermal electrons. In weakly relativistic plasma, the dynamics of one-dimensional IASWs are described by the fractional continuity and motion equations for a normalized fluid. Additionally, closure for the system is provided by the fractional Poisson equation formulated in a one-dimensional fractional representation

$$D_t^\alpha n_i + D_x^\alpha (n_i u) = 0, \quad (1)$$

$$D_t^\alpha (\gamma u) + u D_x^\alpha (\gamma u) + D_x^\alpha \varphi + \delta n_i^{-1} D_x^\alpha p_i = 0, \quad (2)$$

$$D_t^\alpha p_i + u\, D_x^\alpha p_i + 3 p_i D_x^\alpha (\gamma u) = 0, \quad (3)$$

$$D_x^{\alpha\alpha} \varphi = \Omega n_e - (\Omega - 1) n_p - n_i . \quad (4)$$

Note that equations (1) to (4) reduced to the well-known equations as obtained in [44]. In this case, n_i represents the concentration of ions normalized by the unperturbed electron concentration n_{e0}, the electrostatic potential is φ, the ions flow velocities along the x direction is represented by u normalized by $c_s=\sqrt{T_e/m_i}$, T_i, T_p and T_e are the temperature ion, positron and electron plasma, the particles' masses are m_e for electrons and m_i for ions, P_i is pressure and D_x^{α} describes the conformable fractional differential in relation to x, $D_x^{\alpha\alpha}=D_x^{\alpha}D_x^{\alpha}$ the twice conformable fractional differential in relation to x. Ions are thought to have a relatively small relativistic influence, which can be expanded to $\gamma=1/\sqrt{1-u^2/c^2}\approx 1+u^2/2c^2$.

Through the application of reductive perturbation techniques, the scale's new stretching coordinates (time and space) are provided as

$$\xi=\frac{\sqrt{\varepsilon}}{\alpha}\left(x^{\alpha}-\alpha V\, t^{\alpha}\right), \qquad \tau=\frac{\sqrt{\varepsilon^3}}{\alpha}t^{\alpha}, \tag{5}$$

where ε, is the small expansion parameter, which is proportional to the perturbation's amplitude, serves as an indicator of the system's nonlinearity strength. Here, V denotes the fractional spatial phase velocity of the wave propagating along the x-direction. The fractional operator can be represented in the following form

$$D_x^{\alpha}=\sqrt{\varepsilon}\,D_{\xi}^{\alpha}, \quad D_x^{\alpha\alpha}=\varepsilon\,D_{\xi}^{\alpha\alpha}, \quad D_t^{\alpha}=-V\sqrt{\varepsilon}\,D_{\xi}^{\alpha}+\sqrt{\varepsilon^3}\,D_{\tau}^{\alpha}. \tag{6}$$

Substituting the operators in equation (6) into equations (1), (2) and (4) we have

$$-\alpha V\sqrt{\varepsilon}\,D_{\xi}^{\alpha}n_i+\sqrt{\varepsilon^3}\,D_{\tau}^{\alpha}n_i+\sqrt{\varepsilon}\,D_{\xi}^{\alpha}(n_i u)=0, \tag{7}$$

$$-\alpha V\sqrt{\varepsilon}\,D_{\xi}^{\alpha}(\gamma u)+\sqrt{\varepsilon^3}\,D_{\tau}^{\alpha}(\gamma u)+u\sqrt{\varepsilon}\,D_{\xi}^{\alpha}(\gamma u)+\sqrt{\varepsilon}\,D_{\xi}^{\alpha}\varphi+\delta n_i^{-1}\sqrt{\varepsilon}\,D_{\xi}^{\alpha}p_i=0, \tag{8}$$

$$-\alpha V\sqrt{\varepsilon}\,D_{\xi}^{\alpha}p_i+\sqrt{\varepsilon^3}\,D_{\tau}^{\alpha}p_i+u\sqrt{\varepsilon}\,D_{\xi}^{\alpha}p_i+3p_i\sqrt{\varepsilon}\,D_{\xi}^{\alpha}(\gamma u)=0, \tag{9}$$

$$\varepsilon D_{\xi}^{\alpha\alpha}\varphi=\Omega\left[1+(q-1)\varphi\right]^{\frac{q+1}{2(q-1)}}-(\Omega-1)\left[1-\sigma(q-1)\varphi\right]^{\frac{q+1}{2(q-1)}}-n_i. \tag{10}$$

The dependent variables n_i, p_i u and φ can be expanded in the following manner within the power series of ε as

$$\begin{aligned} n_i&=1+\varepsilon n_{i1}+\varepsilon^2 n_{i2}+\varepsilon^3 n_{i3}+\ldots,\\ p_i&=1+\varepsilon p_{i1}+\varepsilon^2 p_{i2}+\varepsilon^3 p_{i3}+\ldots,\\ u&=u_0+\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+\ldots,\\ \varphi&=0+\varphi_1\varepsilon+\varphi_2\varepsilon^2+\varphi_3\varepsilon^3+\ldots, \end{aligned} \tag{11}$$

Since $\gamma=1+u^2/2c^2$, so equations (8) and (9) can be written in this form

$$-\alpha V\sqrt{\varepsilon}D_\xi^\alpha(u+u^3/2c^2)+\sqrt{\varepsilon^3}D_\tau^\alpha(u+u^3/2c^2)+u\sqrt{\varepsilon}D_\xi^\alpha(u+u^3/2c^2)$$
$$+\sqrt{\varepsilon}D_\xi^\alpha\varphi+\delta n_i^{-1}\sqrt{\varepsilon}D_\xi^\alpha p_i=0, \quad (12)$$

$$-\alpha V\sqrt{\varepsilon}D_\xi^\alpha p_i+\sqrt{\varepsilon^3}D_\tau^\alpha p_i+u\sqrt{\varepsilon}D_\xi^\alpha p_i+3p_i\sqrt{\varepsilon}D_\xi^\alpha(u+u^3/2c^2)=0. \quad (13)$$

Using (11) in equation (7) and collecting terms of ε in the lowest order, from equation (7), one can obtain:

$$-\alpha V\sqrt{\varepsilon}D_\xi^\alpha\left(1+\varepsilon n_{i1}+\varepsilon^2 n_{i2}+\varepsilon^3 n_{i3}+...\right)+\sqrt{\varepsilon^3}D_\tau^\alpha\left(1+\varepsilon n_{i1}+\varepsilon^2 n_{i2}+\varepsilon^3 n_{i3}+...\right)$$
$$+\sqrt{\varepsilon}D_\xi^\alpha\left(\left(1+\varepsilon n_{i1}+\varepsilon^2 n_{i2}+\varepsilon^3 n_{i3}+...\right)\left(u_0+\varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+...\right)\right)=0,$$

$$\sqrt{\varepsilon}: \quad D_\xi^\alpha u_0=0,$$

$$\sqrt{\varepsilon^3}: -\alpha V D_\xi^\alpha n_{i1}+u_{i0}D_\xi^\alpha n_{i1}+u_{i1}D_\xi^\alpha n_{i0}+D_\xi^\alpha u_1=0,$$

$$\sqrt{\varepsilon^5}: -\alpha V D_\xi^\alpha n_{i2}+D_\tau^\alpha n_{i1}+u_0 D_\xi^\alpha n_{i2}+u_2 D_\xi^\alpha n_{i0}+D_\xi^\alpha u_{i2}+u_1 D_\xi^\alpha n_{i1}+n_{i1}D_\xi^\alpha u_1=0,$$

Using (11) in equation (12) and collecting terms of ε in the lowest order, from equation (12), one can obtain:

$$-\alpha V\sqrt{\varepsilon}D_\xi^\alpha(u_0+\varepsilon u_1+\varepsilon^2 u_2+...)-\frac{3\alpha V\sqrt{\varepsilon}}{2c^2}(u_0+\varepsilon u_1+...)^2 D_\xi^\alpha(u_0+\varepsilon u_1+...)$$
$$+\sqrt{\varepsilon^3}D_\tau^\alpha(u_0+\varepsilon u_1+\varepsilon^2 u_2+...)+\frac{3\sqrt{\varepsilon^3}}{2c^2}(u_0+\varepsilon u_1+...)^2 D_\tau^\alpha(u_0+\varepsilon u_1+\varepsilon^2 u_2+...)$$
$$+\sqrt{\varepsilon}(u_0+\varepsilon u_1+...)D_\xi^\alpha(u_0+\varepsilon u_1+\varepsilon^2 u_2+...)$$
$$+\frac{3\sqrt{\varepsilon}}{2c^2}(u_0+\varepsilon u_1+...)^3 D_\xi^\alpha(u_0+\varepsilon u_1+\varepsilon^2 u_2+...)+\sqrt{\varepsilon}D_\xi^\alpha\left(\varphi_1\varepsilon+\varphi_2\varepsilon^2+...\right)$$
$$+\delta\sqrt{\varepsilon}\left(1+\varepsilon n_{i1}+\varepsilon^2 n_{i2}+...\right)^{-1}D_\xi^\alpha\left(1+\varepsilon p_{i1}+\varepsilon^2 p_{i2}+\varepsilon^3 p_{i3}+...\right)=0,$$

$$\sqrt{\varepsilon}: \quad -\alpha V D_\xi^\alpha u_0-\frac{3\alpha V u_0^{\,2}}{2c^2}D_\xi^\alpha u_0+u_0 D_\xi^\alpha u_0+\frac{3u_0^{\,2}}{2c^2}D_\xi^\alpha u_0=0,\Rightarrow D_\xi^\alpha u_0=0,$$

$$\sqrt{\varepsilon^3}: \quad -\gamma_1\left(\alpha V-u_0\right)D_\xi^\alpha u_1+D_\xi^\alpha\varphi_1+\delta D_\xi^\alpha p_{i1}=0, \qquad \gamma_1=1+\frac{3u_0^{\,2}}{2c^2},$$

$$\sqrt{\varepsilon^5}: \quad -\gamma_1\left(\alpha V-u_0\right)D_\xi^\alpha u_2+\gamma_1 D_\xi^\alpha u_1+\left(\gamma_1-\frac{2\gamma_2\left(\alpha V-u_0\right)}{u_0}\right)u_1 D_\xi^\alpha u_1+D_\xi^\alpha\varphi_2-\delta n_{i1}D_\xi^\alpha p_{i1}$$
$$+\delta D_\xi^\alpha p_{i2}=0, \qquad \gamma_2=1.5\beta^2, \quad \beta=\frac{u_0}{c}$$

Using (11) in equation (13) and collecting terms of ε in the lowest order, from equation (13), one can obtain:

$$-\alpha V\sqrt{\varepsilon}D_{\xi}^{\alpha}\left(1+\varepsilon p_{i1}+\varepsilon^{2}p_{i2}+...\right)+\sqrt{\varepsilon^{3}}D_{\tau}^{\alpha}\left(1+\varepsilon p_{i1}+\varepsilon^{2}p_{i2}+...\right)$$
$$+\sqrt{\varepsilon}(u_0+\varepsilon u_1+...)D_{\xi}^{\alpha}\left(1+\varepsilon p_{i1}+...\right)+3\sqrt{\varepsilon}\left(1+\varepsilon p_{i1}+...\right)D_{\xi}^{\alpha}(u_0+\varepsilon u_1+...)$$
$$+\frac{9\sqrt{\varepsilon}}{2c^2}\left(1+\varepsilon p_{i1}+...\right)(u_0+\varepsilon u_1+...)^2D_{\xi}^{\alpha}(u_0+\varepsilon u_1+...)=0,$$
$$\sqrt{\varepsilon^3}:\ -(\alpha V-u_0)D_{\xi}^{\alpha}p_{i1}+3\gamma_1D_{\xi}^{\alpha}u_1=0,$$
$$\sqrt{\varepsilon^5}:\ D_{\tau}^{\alpha}p_{i1}-(\alpha V-u_0)D_{\xi}^{\alpha}p_{i2}+3\gamma_1D_{\xi}^{\alpha}u_2+u_1D_{\xi}^{\alpha}p_{i1}+3\gamma_1p_{i1}D_{\xi}^{\alpha}u_1$$
$$+\frac{6\gamma_2}{u_0}u_1D_{\xi}^{\alpha}u_1=0,$$

By substituting equation (11) into equation (10) and collecting terms of the lowest order, one can derive the following expression from equation (10):

$$\varepsilon D_{\xi}^{\alpha\alpha}\left(\varphi_1\varepsilon+\varphi_2\varepsilon^2+...\right)=\Omega(1+\frac{1}{2}(q+1)(\varphi_1\varepsilon+\varphi_2\varepsilon^2+...)$$
$$-\frac{1}{8}(q+1)(q-3)(\varphi_1\varepsilon+\varphi_2\varepsilon^2+...)^2+...)-(\Omega-1)(1-\frac{\sigma}{2}(q+1)(\varphi_1\varepsilon+\varphi_2\varepsilon^2+...)$$
$$-\frac{\sigma^2}{8}(q+1)(q-3)(\varphi_1\varepsilon+\varphi_2\varepsilon^2+...)^2-...)-(1+\varepsilon n_{i1}+\varepsilon^2n_{i2}+\varepsilon^3n_{i3}+...),$$
$$\varepsilon^0:\qquad \Omega-(\Omega-1)-1=0,$$
$$\varepsilon:\qquad \frac{1}{2}(q+1)\Omega\varphi_1+\frac{\sigma}{2}(q+1)(\Omega-1)\varphi_1-n_{i1}=0,$$
$$\varepsilon^2:\qquad D_{\xi}^{\alpha\alpha}\varphi_1=\frac{(q+1)(1+a\sigma)}{2(1-a)}\varphi_2-\frac{(q+1)(q-3)(1-a\sigma^2)}{8(1-a)}\varphi_1^2-n_{i2}.$$

Doing the conformable factional integration and using the boundary conditions,

$$n_{i1}=0,\qquad v_1=0,\quad p_{i1}=0,\quad u_1=0,\quad \varphi_1=0\quad \text{at}\quad \xi\to\infty, \tag{14}$$

to obtain the subsequent perturbed first-order quantities:

$$n_{i1}=\frac{1}{\gamma_1[(\alpha V-u_0)^2-3\delta]}\varphi_1=\frac{(q+1)(1+a\sigma)}{2(1-a)}\varphi_1,$$
$$u_1=-\frac{\alpha V-u_0}{\gamma_1[(\alpha V-u_0)^2-3\delta]}\varphi_1, \tag{15}$$
$$p_{i1}=\frac{3}{(\alpha V-u_0)^2-3\delta}\varphi_1,$$

Also, we get the (1+1) nonlinear fractional KdV model may be created

$$D_{\tau}^{\alpha}\varphi_1+A_1\varphi_1D_{\xi}^{\alpha}\varphi_1+A_2D_{\xi}^{\alpha\alpha\alpha}\varphi_1=0, \tag{16}$$

The formula mentioned above represents the well-known one-dimensional space-time fractional KdV equation, which is highly valuable for studying the nonlinear propagation of ion-acoustic shock structures within the plasma system under investigation. The following forms are derived for equation (16) the nonlinearity A_1, and dispersion A_2:

$$A_1 = \frac{1}{2K\gamma_1^2}\left(\gamma_1(\alpha V - u_0) + \frac{2\gamma_2(\alpha V - u_0)^2}{u_0} + \frac{6\delta\gamma_2}{u_0}\right) + \frac{(q-3)(1-a\sigma^2)K}{4(\alpha V - u_0)(1+a\sigma)} + \frac{1}{\alpha V - u_0}\left[\frac{9\delta}{2K} + \frac{1}{\gamma_1}\right], \tag{17}$$

$$A_2 = \frac{\gamma_1 K^2}{2(\alpha V - u_0)}, \qquad K = (\alpha V - u_0)^2 - 3\delta. \tag{18}$$

From first equation of (15) we have

$$\frac{1}{\gamma_1[(\alpha V - u_0)^2 - 3\delta]} = \frac{(q+1)(1+a\sigma)}{2(1-a)}.$$

So the fractional phase velocity takes the formulae

$$V = \frac{1}{\alpha}\left[u_0 + \sqrt{3\delta + \frac{2(1-a)}{\gamma_1(q+1)(1+a\sigma)}}\right], \tag{19}$$

from the explicit formulae of the phase velocity *V*, we find an inverse relation between *V* and the fractional order α this means that when α increases the phase velocity *V* decreases and vice versa.

We use the F-expansion approach to create exact analytic solutions for the fractional space-time KdV model. The F-expansion approach is an efficient and straightforward algebraic technique for determining the precise solutions to nonlinear evolution problems [Wang M. L. and Li X. Z., 2005] , [Li W.-W., Tian Y., and Zhang Z.2012], which has been used to solve several nonlinear equations. Consider the fractional space-time KdV equation in the current form, as indicated in Equation (16). We apply the traveling wave transformation as $\varphi_1(\xi^\alpha, \tau^\alpha) = \varphi_1(\zeta)$, and $\zeta = (k\xi^\alpha - \alpha\omega\tau^\alpha)/\alpha,$. Thus, Equation (16) is converted into the following ODE

$$-\alpha\omega\frac{d\varphi_1}{d\zeta} + kA_1\varphi_1\frac{d\varphi_1}{d\zeta} + k^3A_2\frac{d^3\varphi_1}{d\zeta^3} = 0, \tag{20}$$

The fractional space-time KdV model can be solved by using the F-expansion method as

$$\varphi_1(\zeta) = a_0 + \sum_{j=1}^{N} a_j F^j(\zeta), \qquad a_N \neq 0, \quad F' = \sqrt{A + BF^2 + CF^4}, \tag{21}$$

with $a_0, a_1, ..., a_N$ are arbitrary constants in this case. *N= 2* can be obtained by balancing the highest-order linear partial derivative term and the highest-order nonlinear term in Equation (20). Equation (20) has a solution that looks like this

$$\varphi_1(\zeta) = a_0 + a_1 F(\zeta) + a_2 F^2(\zeta), \tag{22}$$

From equation (21), we have

$$\begin{aligned} \frac{d\varphi_1}{d\zeta} &= (a_1 + a_2 F)\sqrt{A + B\,F^2 + C\,F^4}, \\ \varphi_1 \frac{d\varphi_1}{d\zeta} &= \left(a_0 a_1 + (a_0 a_2 + a_1^2)F + 2a_1 a_2 F^2 + a_2^2 F^3\right)\sqrt{A + B\,F^2 + C\,F^4}, \\ \frac{d^3\varphi_1}{d\zeta^3} &= \left(a_1 B + 4a_2 BF + 6a_1 CF^2 + 12a_2 CF^3\right)\sqrt{A + B\,F^2 + C\,F^4}, \end{aligned} \tag{23}$$

Substituting equations (23) to (20), we have

$$\begin{aligned} &\left[-\alpha\,\omega\left(a_1 + a_2 F\right) + kA_1\left(a_0 a_1 + (a_0 a_2 + a_1^2)F + 2a_1 a_2 F^2 + a_2^2 F^3\right)\right. \\ &\left.+k^3 A_2\,\left(a_1 B + 4a_2 BF + 6a_1 CF^2 + 12a_2 CF^3\right)\right]\sqrt{A + B\,F^2 + C\,F^4} = 0, \end{aligned} \tag{24}$$

Collecting the coefficients of $F(\zeta),\ i = 0,1,\ldots$ and setting the coefficients equal to zero, we have the following system of algebraic Equation

$$\begin{aligned} F^0:&\quad a_1[-\alpha\,\omega + k\,A_1\,a_0 + A_2\,k^3 B] = 0, \\ F^1:&\quad k\,A_1\,a_1^2 + a_2[-\alpha\omega + k\,A_1\,a_0 + 4A_2 k^3 B] = 0, \\ F^2:&\quad 2a_1[k\,A_1\,a_2 + 3A_2 k^3 C] = 0, \\ F^3:&\quad a_2 k\,[A_1\,a_2 + 12A_2 k^2 C] = 0. \end{aligned} \tag{25}$$

Solving this system, we obtain the following solution for the parameters ω, a_0, a_1, a_2 and k as

$$a_0 = -\frac{4A_2 k^3 B - \alpha\omega}{A_1\,k}, \qquad a_1 = 0, \qquad a_2 = -\frac{12A_2 k^2 C}{A_1}. \tag{26}$$

The electrostatic potential can be determined using IASW solution by substituting from Equation (26) into (22); we have

$$\varphi_1(\zeta) = -\frac{4A_2 k^3 B - \alpha\omega}{A_1\,k} - \frac{12A_2 k^2 C}{A_1} F(\zeta)^2, \qquad \zeta = \frac{k\,\xi^\alpha - \alpha\omega\tau^\alpha}{\alpha}. \tag{27}$$

Equation (27) is the general solution depending on choosing the parameters *A*, *B*, *C*, and the corresponding function *F;* we recommended reference [Wang M.. L. and Li X.Z.2005], for more information about the *F* expansion method. Using this technique, we can obtain the general exact solution, including single and combined Jacobi elliptic function solutions, soliton-like solutions, solitary wave, and trigonometric function solutions.

Case 1: when $A = 1,\ B = -(1+m^2),\ C = m^2$, and $F(\zeta) = \mathrm{sn}(\zeta, m)$

$$\varphi_{11}(\zeta) = \frac{4A_2 k^3 (1+m^2) + \alpha\omega}{A_1\,k} - \frac{12A_2 k^2 m^2}{A_1}\left[\mathrm{sn}(\zeta, m)\right]^2, \qquad \zeta = \frac{k\,\xi^\alpha - \alpha\omega\tau^\alpha}{\alpha}. \tag{28}$$

Case 2: when $A=1-m^2,\ B=2m^2-1,\ C=-m^2$, and $F(\zeta)=\mathrm{cn}(\zeta,m)$

$$\varphi_{12}(\zeta)=-\frac{4A_2k^3(2m^2-1)-\alpha\omega}{A_1k}+\frac{12A_2k^2m^2}{A_1}\left[\mathrm{cn}(\zeta,m)\right]^2,\qquad \zeta=\frac{k\,\xi^\alpha-\alpha\omega\tau^\alpha}{\alpha}. \quad (29)$$

Case 3: when $A=m^2-1,\ B=2-m^2,\ C=-1$, and $F(\zeta)=\mathrm{dn}(\zeta,m)$

$$\varphi_{13}(\zeta)=-\frac{4A_2k^3(2-m^2)-\alpha\omega}{A_1k}+\frac{12A_2k^2}{A_1}\left[\mathrm{dn}(\zeta,m)\right]^2,\qquad \zeta=\frac{k\,\xi^\alpha-\alpha\omega\tau^\alpha}{\alpha}. \quad (30)$$

Case 4: when $A=1,\ B=-2,\ C=1$, and $F(\zeta)=\tanh(\zeta)$

$$\varphi_{14}(\zeta)=\frac{8A_2k^3+\alpha\omega}{A_1k}-\frac{12A_2k^2}{A_1}\left[\tanh(\zeta)\right]^2,\qquad \zeta=\frac{k\,\xi^\alpha-\alpha\omega\tau^\alpha}{\alpha}. \quad (31)$$

Case 5: when $A=0,\ B=1,\ C=-1$, and $F(\zeta)=\mathrm{sech}(\zeta)$

$$\varphi_{15}(\zeta)=-\frac{4A_2k^3-\alpha\omega}{A_1k}+\frac{12A_2k^2}{A_1}\left[\mathrm{sech}(\zeta)\right]^2,\qquad \zeta=\frac{k\,\xi^\alpha-\alpha\omega\tau^\alpha}{\alpha}. \quad (32)$$

and so on.

The fact that the fractional KdV equation recovers classical results when $\alpha=1$, validates the theoretical framework for fractional equations and establishes that the fractional approach can extend classical results into more generalized contexts. This extension is particularly valuable in fields such as fluid dynamics, nonlinear optics, and plasma physics, where wave dynamics often exhibit complex behaviors that cannot be fully captured by traditional models. Fractional calculus provides a more flexible and accurate tool for studying such phenomena, offering deeper insights and potential for more precise modeling in these fields, advancing both scientific understanding and technological applications.

The results obtained for correspond to the well-known solutions of the KdV equation. This observation highlights the accuracy and consistency of the methods deriving the fractional KdV equation. Such findings validate the theoretical framework and underscore the applicability of fractional calculus in capturing and extending classical results to scenarios involving fractional orders. This capability is pivotal in various fields, including fluid dynamics, nonlinear optics, and plasma physics, where understanding and manipulating wave dynamics are essential for advancing scientific understanding and technological applications.

3. Summery, discussion and open problems

This study addresses the derivation and physical significance of the one-dimensional nonlinear KdV equation, particularly in the context of dissipation quantum plasma. The emergence of the KdV equation here is tied to the study of shocks and the propagation of nonlinear IASWs within plasma systems, using reductive perturbation theory. This theory is a common analytical technique used to reduce complex nonlinear PDEs to simpler forms, allowing for the study of wave phenomena like shocks and solitons. The KdV equation is of great importance in various physical fields, especially in understanding nonlinear wave propagation in plasma physics, fluid dynamics, and other areas involving wave-like phenomena. In quantum plasmas, where dissipation effects are significant, the KdV equation provides insights into the behavior of nonlinear waves, such as how solitary waves form, propagate, and interact under dissipation.

The F-expansion method, a mathematical technique for solving nonlinear PDEs, was employed to solve this one-dimensional KdV problem, specifically focusing on ions within solitary traveling waves. The F-expansion method allows for finding exact analytical solutions of nonlinear wave equations, which is critical in studying the dynamics of solitary waves in plasma, especially when the system's complexity requires more advanced solution techniques. This method enhances our understanding of how these solitary waves behave, and helps in analyzing their properties, such as speed and amplitude, under varying conditions.

This study discusses the IAWs in complex plasma environments, where solutions to the governing equations are derived, revealing key dependencies on several plasma parameters. These parameters include positron concentration, temperature ratios between electrons and positrons as well as ions and electrons, ion kinematic viscosity, and the nonextensive behavior of electrons and positrons. The inclusion of nonextensivity refers to the departure from traditional thermodynamic equilibrium, where the distribution of particles does not follow a Maxwell-Boltzmann distribution but instead reflects more complex dynamics often seen in plasmas or astrophysical environments. A central focus of the study is the significant influence of fractional order on the phase velocity of IAWs. The phase velocity of these waves is critical in understanding wave propagation in plasmas, as it is affected by factors like the wave's energy, the type of plasma, and the underlying physics governing the system. The fractional order adds a new layer of complexity to the dynamics, potentially offering more accurate descriptions of wave behaviors in environments where traditional integer-order models fall short, such as in complex or highly turbulent plasmas.

The study’s findings shed light on how these intricate plasma parameters, including nonextensivity and the fractional order, impact the propagation of ion acoustic waves. This has important implications for understanding fundamental astrophysical processes, such as those occurring in stellar atmospheres, interstellar media, or other cosmic plasma environments. By incorporating fractional calculus, the study opens up new avenues for analyzing and modeling plasma dynamics in a more comprehensive and nuanced way, ultimately enhancing our understanding of wave phenomena in both terrestrial and astrophysical contexts.

Future studies could apply the derived fractional KdV-like equations to real astrophysical systems such as the interstellar medium, solar winds, or magnetospheres, where complex plasmas are prevalent. Likewise, laboratory experiments simulating space and astrophysical plasmas could help validate the theoretical models and provide direct comparisons with observed data. Further investigation into the nonlinear behavior of ion acoustic waves, especially in the context of fractional plasma models, would be valuable. Nonlinear wave phenomena like wave steepening, shock formation, and soliton interactions could be studied to understand how fractional order affects wave stability and energy transfer in plasmas.

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