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FULL PAPER

Composition of Steenrod Square Operations on Symmetry Cohomomology of Topological Spaces with Applications.

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Abstract:

This paper aims to combine the constructions of classical Steenrod operations, such as homogeneity operations on polynomials on ZP and cohomology of topological spaces, where p is a prime integer. Cohomological processes are natural-to-natural transformations, and then we define the characteristics of Steenrod processes and we define what we are aiming for, so we will proceed with the construction. This will be used space-wise, and then make sure that the build we're doing implements Steenrod operations; in the odd case, there won't be sufficient space in the project for this. We have included several immediate applications, and then we will briefly discuss the build we have completed and propose that further development should continue from this point. Lastly, we have some necessary details and calculations to ensure the smooth functioning of the build in the field.

KEYWORDS: Steen rod square, power operation, vector bundles, characteristic classes, cohomology theory, homology theory, convex function.

Introduction:

In algebraic topology, so-called Steenrod squares are the arrangement of homogeneities on regular cohomology with coefficients in Z_2 that are homogeneous to the suspension ("stable homogeneities"). They are special examples of energy processes. Steenrod processes are power-and-energy processes arising from the cup product's commutative but modest version of commutativity, involving operations that take powers of *pth*.

In 1958, Adams used it to compute sets of stable homogeneous spheres, and in the same year, Milnor proved that Steenerod algebras and dual algebras have structures of Hopf algebras. (Elhamdadi, M. 2003). The Steenrod squares play a crucial role in algebraic topology, particularly in the study of cohomology operations. By understanding their properties and constructions, we can gain deeper insights into the structure of various algebraic objects and their applications in geometric contexts.

Definition 1:

In terms of Steenrod operations we define Stiefel–Whitney characteristic $sw(\xi\ )$ as factors of cohomology in a group G, (hom) of degree i is a morphism

θ: $Hom^{*}(\ \ , G) \rightarrow Hom^{*+i}(\ \ , G)$ an operation from the topological space class to the set class. (Husemoller, D. *et al*., 2008) (122-125). To introduce the Steenrod process we must know Bockstein symmetries as examples of cohomology symmetry operations.

Steenrod square:

For cohomology homogeneous over $\mathbf{Z}_2$ from two components, there is a unique process Stq^{i}: $Hom^{*}(, F_2) \rightarrow Hom^{*+i}(, Z_2)$ of i degree then Stq^{i} navigates with commentary and, for x ∈ $Hom^{i}(X, Z_2)$, $Stq^{i}(x) = x^2$, the cup square. (Husemoller, D. *et al.*, 2008) (123-124). Then operation Stq^{i} is called the Steenrod square

Definition 2:

For X fixed topological space, from sequence

$$0 \rightarrow Z_2 \rightarrow Z_4 \rightarrow Z_2 \rightarrow 0$$

The operation Stq^n are cohomology operations

$$Stq^n : Hom^k (X,\ Z_2) \longrightarrow Hom^{k+n}(X, Z_2)$$

This is also known as the Steenrod square Stq^1, hence of morphisms in the homotopy category. (Stq^1) called the Bockstein homomorphism.

The Stq^n fulfill the following requirements. (Husemoller, D. *et al.*, 2008) (123-124).

(1) In degree 0, Stq^{0} is the identity, and $Stq^{i}|Hom^{n}(\ ,\mathbf{Z}_2) = 0$ for $i>n$.

(2) (Cartan's formula), for $x, y \in Hom^{*}(X, \mathbf{Z}_2)$, then

$$Stq^k(xy) = \sum_{k=i+j} Stq^i(x)Stq^j(y).$$

Multiproduct version is

$$Stq^q(x_1 \dots x_r) = \sum_{i(1)+\dots+i(r)=q} Stq^{i^{(1)}}(x_1)\dots Stq^{i^{(r)}}(x_r).$$

(3) $Stq^i(x+y) = Stq^i(x) + Stq^i(y)$

(4) (Adem's relations), for $0 < m < 2n$, the iterate of Stq^n satisfies

$$Stq^m Stq^n = \sum_{j=0}^{\left[\frac{m}{2}\right]} \binom{n-1-j}{m-2j} Stq^{m+n-j} Stq^{j}.$$

(5) $Stq^i(\sigma(x)) = \sigma\left(Stq^i(x)\right)$, where σ is the suspension map.

When $p = 2$ then $\theta^i = Stq^i$, that gives Steenrod squares Stq^n .

For p *is* odd, then we have $c \cdot \theta_{2i(p-1)} = P^i$ and $c \cdot \theta_{2i(p-1)+1} = \beta P^i$, that gives Steenrod powers, where c is a constant.

For Stq^n on low-dimensional characteristic we have the following theorem.

Theorem 1:

On low-dimensional categories, we have the following Steenrod operations Stq^n:

Consider to one and two dimensions we have.

(1) If $x \in Hom^1(X, \mathbf{Z}_2)$, that we have $Stq^i(x^m) = \binom{m}{1} x^{m+i}$

(2) If $y \in Hom^2(X, \mathbf{Z}_2)$ and if $Stq^1(y) = 0$, then $Stq^{2i}(y^m) = \binom{m}{1} y^{m+i}$

and

$$Stq^{2i+1}(y^m) = 0.$$

Proof: By induction on m, when $m = 0$ is clear.

Case (1) is illustrated as following formula.

$$Stq^i(x^m) = Stq^i(x.x^{m-1}) = Stq^0(x).Stq^i(x^{m-1}) + Stq^1(x).Stq^{i-1}(x^{m-1})$$

$$= \binom{m-1}{i} + \binom{m-1}{i-1} x^{m+i} = \binom{m}{i} x^{m+i}.$$

For cases of Adem's relations we have followed concept:

(1) When $n = 1$, we have $1 \le n$, one terms for $j = 0$, thus, we have sum corresponding,

$$Stq^2 Stq^n = \binom{n-1}{1} Stq^{n+1} = \begin{cases} Stq^{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

with simple formula $Stq^1 Stq^1 = 0$, $Stq^1 Stq^2 = Stq^3$,

$Stq^1 Stq^3 = 0$, and $Stq^1 Stq^4 = Stq^5$. (Husemoller, D. *et al.*, 2008) (125-126).

(2) There are only two terms within the sum of the two, according to

$i = 0$ and $i = 1$, for n = 2. This is the case that $2 \leq n$. So, in this instance, we have

$$Stq^2 Stq^n = \binom{n-1}{2} Stq^{n+2} + \binom{n-2}{0} Stq^{n+1} Stq^1.$$

This splits into two cases focusing on *n* mod 4.

$$Stq^2 Stq^n = Stq^{n+1} Stq^1 + \begin{cases} Stq^{n+2} & \text{for } n \equiv 0,\ 3(\text{mod}4) \\ 0 & \text{for } n \equiv 1,2(\text{mod}4) \end{cases}$$

with simple cases $Stq^2 Stq^2 = Stq^3 Stq^1$, $Stq^2 Stq^3 = Stq^4 Stq^1 + Stq^5$,

$Stq^2 Stq^4 = Stq^5 Stq^1 + Stq^6$, $Stq^2 Stq^5 = Stq^6 Stq^1$, and $Stq^2 Stq^7 = Stq^8 Stq^1 + Stq^9$.

On integer **Z**, the induce effect of binomial $\binom{n}{i}$ is the efect of x^i in the polynomial

$(1 + x)^n \in Z[x]$. Here, integers defined within the modulos of 2

Example (Equivalencies of Two Mod 2):

Forfield $\mathbf{Z}_2 = \{0, 1\}$ of two elements for $n \in \mathbf{Z}$. (Husemoller, D. *et al.*, 2008).

$$\binom{n}{1} = \begin{cases} 0 & if\ n\ \ is\ \ even \\ 1 & if\ n\ \ is\ \ odd \end{cases}$$

and

$$\binom{n}{2} = \begin{cases} 0 & if\ n = 0,1(mod\ 4) \\ 1 & if\ n = 2,3(mod\ 4) \end{cases}$$

Definition 3:

A bundle ξ, represented by ξ_B, comprises a Thom space that corresponds to the divide bundle Dis(ξ)/Sp(ξ).

Then the map σ: $Hom^{i+n}(Dis(\xi)/Sp(\xi)) \rightarrow Hom^{i+n}(\xi_B)$ is symmetric, and then the Thom map is defined as ψ: $Hom^i(B) \rightarrow \overline{Hom}^{i+n}(\xi_B)$ to be $\psi = \sigma\phi'$. (Husemoller, D. 1994)

Theorem 2:

By cohomology characteristic $U_\xi \in Hom^n(Dis(\xi)/Sp(\xi))$ and the complete Steenrod process $Stq = \sum_{0\leq i} Stq^i$. (Husemoller, D. *et al.*, 2008) (132-133), we generate a complete Stiefel-Whitney characteristic $Stq(U_\xi) = sw(\xi)U_\xi$ or $sw(\xi) = \varphi^{-1}(Stq(U_\xi))$.

Dis(ξ) is a bundle of disks, and Sp(ξ) is a bundle of spheres.

Proof: By the splitting principal bundle, we can check a formula by doing it only for

$\xi = L_1 \oplus \ldots \oplus L_n$, a sum of line bundles, we have a cup product

decomposition of $U_\xi = U_1 \ldots U_n$ of 1-dimensional character U_i related to L_i. Only Stq^1 is nonnull on Ui, and it is $\mathrm{Stq}^1(\mathrm{U_i}) = \mathrm{U_i^2}$. Hence, through Cartan's formula of multiproduct, we have the computation bellow

$$\mathrm{Stq}^r(\mathrm{U}_\xi) = \mathrm{Stq}^r(\mathrm{U}_1 \dots \mathrm{U}_n) = \sum_{i(1)<\dots<i(r)} \mathrm{U}_1 \dots \mathrm{U}_{i(1)}^2 \dots \mathrm{U}_{i(r)}^2 \dots \mathrm{U}_n$$

$$= \sum_{i(1)<\cdots<i(r)} \mathrm{U}_{i(1)}^2 \dots \mathrm{U}_{i(r)}^2 (\mathrm{U}_1 \dots \mathrm{U}_n) = \mathrm{sw}_r(\mathrm{L}_1 \oplus \dots \oplus \mathrm{L}_n)(\mathrm{U}_1 \dots \mathrm{U}_n)$$

via the splitting linking to cohomology, we see

$Stq^r(U_\xi) = sw_r(\xi) U_\xi$. This completes the theorem. (Husemoller, D. *et al.*, 2008) (132-133).

Definition 4 *(Thom):*

The Stiefel- Whitney characteristic $sw_i(\xi)$ is denoted by $\phi^{-1}(Stq^i U_\xi)$.

$$sw_i(\xi) = \phi^{-1}(Stq^i U_\xi). \text{ (Marathe, K. 2010).}$$

Where ϕ called Thom isomorphism

Equivalently, $sw_i(\xi)$ is the characteristic that $\phi(sw_i(\xi)) = Stq^i\ \phi\ (1)$, In terms of the complete square *Stq*, the complete Stiefel-Whitney characteristic

$sw(\xi) = sw_0(\xi) + sw_1(\xi) + \dots$ is given by $\phi^{-1}(Stq^i)\ \phi\ (1)$.

Theorem 3:

The Euler characteristic, denoted by $eu(\xi)$, is held by the natural symmetric

$\mathrm{Hom}^n(B; Z) \rightarrow \mathrm{Hom}^n(B; Z_2)$ to the upper Stiefel-Whitney characteristic, $sw_n(\xi)$. (Giansiracusa, J. *et al.*, 2003).

Proof. It is obvious that the cohomology characteristic μ element corresponds to the mod2 cohomology characteristic μ and $\mu \sqcup \mu$ connected to $Stq^n(\mu)$ if we apply surjection factor $Z \rightarrow Z_2$ to both sides of $eu(\xi) = \varphi^{-1}(\mu \sqcup \mu)$. Thus, $\varphi^{-1}(\mu \sqcup \mu)$ is associated with $\varphi^{-1}\mathrm{Stq}^n(\mu) = \mathrm{sw}_n(\mu)$.

The natural symmetric $Hom^n(B; \mathbf{Z}) \rightarrow Hom^n(B; \mathbf{Z}_2)$ holds the Euler characteristic denotes by $eu(\xi)$ to upper Stiefel-Whitney characteristic $sw_n(\xi)$. (Giansiracusa, J. *et al.*, 2003).

If we apply surjection factor $\mathbf{Z} \rightarrow \mathbf{Z}_2$ to two sides of $eu(\xi) = \varphi^{-1}(\mu \sqcup \mu)$ then with clear proof the element of cohomology characteristic μ correspond to the mod2 cohomology characteristic μ and $\mu \sqcup \mu$ related to $Stq^n(\mu)$. Hence $\varphi^{-1}(\mu \sqcup \mu)$ related to $\varphi^{-1}Stq^n(\mu) = sw_n(\mu)$.

Definition 5(Poincaré duality):

For a compact m-dimensional manifold M, for $\delta \in Hom^r(M)$ and for

$\gamma \in Hom^{m-r}(M)$. When $\delta \times \gamma$ is an element, we can define a direct product

$\langle \cdot \rangle : Hom^r(M) \times Hom^{m-r}(M) \rightarrow \mathbf{R}$ by

$$\langle \delta, \gamma \rangle = \int_M \delta \times \gamma \qquad (1)$$

The direct product is a bilinear operation. Additionally, it is non-monogamous, meaning that if $\delta = 0$ or $\gamma \neq 0$, the pairing $\langle \delta, \gamma \rangle$ cannot vanish in a similar manner. Thus, equation (1) define as a dual of $Hom^r(M)$ and

$Hom^{m-r}(M)$, $Hom^r(M) \cong Hom^{m-r}(M)$called the Poincaré duality. (Nakahara, M.2002).

Manifold for the Stiefel-Whitney characteristic in terms of Wu's Formula:

The Steenrod squares $Stq = \sum_i Stq^i$ and the Stiefel–Whitney characteristic $sw(\xi)$ of a bundles are connected by their form $sw(\xi) = \varphi^{-1}(Stq(U_\xi))$. Using Poincare's dualism and its relation to U_M, we derive the Wu characteristic and its relation to the Stiefel-Whitney characteristic of the bundle of tangents. (Giansiracusa, J. 2003; Milnor, J.W. 1981).

***Corollary* 1:**

Let $Stq^{tr}: Hom(X) \rightarrow Hom(X)$ the complete Steenrod square is the transpose of $Stq{:}Hom^*(X) \rightarrow Hom^*(X)$. In specially case, we take $Stq(n), m = n, Stq^{tr}(m)$ *for* $n \in Hom^*(X)$, $m \in Hom^*(X)$. (Husemoller, D. *et al.*, 2008).

Definition 6:

Let $\bar{M}$ be a closed manifold with a Poincare´ duality isomorphism.

$D{:}\ Hom^i(\bar{M}) \rightarrow Hom^{n-i}(\bar{M})$, and let $[M]$ be fundamental characteristic. The Wu characteristic of $\bar{M}$ *is*

$$V = D^{-1}(Stq^{tr}([M])).$$ (Husemoller, D. *et al*, 2008; Husemoller, D. 1994).

The characteristic of the Wu characteristics is that

$\langle n, D(v)\rangle = \langle n, Stq^{tr}([M])\rangle = \langle Stq(n), [M]\rangle = \langle n, v \frown [M]\rangle = \langle nv, [M]\rangle$. $(\frown)$ means cap product.

Theorem 4:

Let $\bar{M}$ be a closed differentiable manifold. Then, the Stiefel-Whitney characteristic $sw(\bar{M}) = sw(T(\bar{M}))$ the bundle of tangents is provided by the Wu characteristic's Steenrod square:$sw(\bar{M}) = Stq(v)$. (Husemoller, D. *et al.*, 2008) (132-133).

Proof: *see* (Husemoller, D. 1994) (275-276).

Association to Bockstein homomorphism

If Stq^1 is the Bockstein homomorphism of the short exact sequence $Z_2 \rightarrow Z_4 \rightarrow Z_2$.

The Steenrod squares are harmonic with comment symmetry. Therefore, Steenrod squares are also known as stable cohomology operations.

Association to Massey products

We see that Massey product relation to Steenrod squares *stq*.

Let $\omega, \omega_1, \omega_2 \in Hom^*(X, Z_2)$ such that their triple Massey product exists. Then the cup product of ω with the triple Massey product is independent of the ambiguity in the Massey products and equals the cup product of ω_1 with ω_2 and with the Steenrod square of ω of degree $deg(\omega) - 1$

$$\omega \smile \langle \omega_1, \omega, \omega_2 \rangle = \omega_1 \smile \omega_2 \smile Stq^{deg(\omega)-1}(\omega).$$

Pontrjagin characteristic

Definition 7:

The *nth* Pontrjagin characteristic of bundle ξ, the real vector bundle, is represented by $P_n(\xi)$, is $(-1)^{n}c_{2n}$ $(\xi \oplus \mathbf{C})$. Where $P(\xi)$ is a member of $Hom^{4n}(B, \mathbf{Z})$. (Husemoller, D. 1994); Giansiracusa, J. *et al*., 2003). We explain $P(\xi) = 1 + P_1(\xi) + \ldots \in Hom^*(B, \mathbf{Z})$ to be the complete Pontrjagin characteristic of real vector bundle ξ are generated, over the algebra of Steenrod powers, by those of the form P_{q^n}.

The Whitney sum proposition can only be satisfied as shown below:

$$2(p(\xi \oplus \eta) - p(\xi)p(\eta)) = 0$$

Let q: $\mathbf{R}P^{2n-1} \rightarrow \mathbf{C}P^{n-1}$. Each real line specified by $z \in Sp^{2n-1}$ can be assigned an isomorphism where {x, -x} determines the complex vector bundle.

By definition, we have $P_n(\xi_{\mathbf{R}}) = (-1)^{n}c_{2n}(\xi \oplus \mathbf{C})$ where$(\xi \oplus \mathbf{C})$ is complex be converted to a real by an equation $P_1(\xi) = ch_1(\xi)^2 - 2ch_2(\xi)$, where *ch* means chern characteristic. (Husemoller, D. *et al*, 2008); Milnor, J.W. 1981).

Theorem 5:

For any smooth even-vector bundle ξ, $p_n(\xi) = eu(\xi)^2$. (Milnor, J.W. 1981).

Proof: $P_n(\xi) = (-1)^{n}c_{2n}(\xi_{\mathbf{C}}) = (-1)^n eu(\xi_{CR}) = \pm eu(\xi^n \oplus \xi^n) = (-1)^n eu(\xi)^2$.

Instead of using the curvature two-form, it is frequently represented as p(M) in relation to the tangent bundle. Since $p_0(M) = 1$ denotes each Pontrjagin characteristic, $p_2(M)$ disappears as a differential form. (Nakahara, M. 2002).

***Lemma* 2:**

The Pontrjagin characteristic of a complex vector bundle (ξ) with n dimensions is determined by the Chern characteristic (Ch) via an equation. (Cohen, R. L. 1998); Janis, L. 2014).

$$1 - P_1(\xi) + P_2(\xi) - \ldots = (1 - ch_1(\xi) + ch_2(\xi) - \ldots)(1 + ch_1(\xi) + ch_2(\xi) + \ldots).$$

Proof: ch(ξ⊕$_{\mathbf{R}}$**C**) = *c*h(ξ)*ch*(ξ*) =$\sum_{i=0}^{\infty} ch_i(\xi) \sum_{i=0}^{\infty} (-1)^i ch_i(\xi)$. Moreover, if k=1(mod2) then *ch*$_k$(ξ ⊕**C**)= $\sum_{0\le i\le k} (-1)^i ch_i(\xi)$ $ch_{k-1}(\xi)$ =0. So, the sum of all even Chern characteristics is the total of every Chern characteristic. (Janis, L. 2014). Pontrjagin characteristic and Squares of odd-dimensional Steifel-Whitney characteristic we have the Wu characteristic in degrees $2i$ and $(sw_i)^2$ not an analysis Stiefel-Whitney characteristic homogeneous to Wu characteristic, in lower degrees. The most notable works in an analysis term not involving sw_1 but they are relating with squares mentioned above.

For odd values of i, we utilize the torsion Pontrjagin characteristic. (Hisham, S. 2011) to achieve this. These characteristics are denoted as P_{4k+2} with an index such that $P_{4i} = p_i$, which aligns with the standard ith Pontrjagin characteristics. In cases where the degrees are $4k + 2$ and involve 2-torsion, we have $2P_{4k+2} = 0 \in Hom^{4k+2}\mathbf{Z}$. By utilizing the Bockstein and Steenrod squares on the Stiefel-Whitney characteristic at degree $2k + 1$, one can establish these properties for a vector bundle E.

Then $P_{4k+2}(E) = \beta Stq^{2k} sw_{2k+1}(E)$.

The mod2 reduction $\rho_2: Hom^{4k+2}(X; \boldsymbol{Z}) \to Hom^{4k+2}(X; \boldsymbol{Z_2})$ of these characteristics give exactly the required squares of Stiefel-Whitney characteristic $\rho_2: P_{4k+2}(E) = sw_{2k+1}(E)^2$. As a result, we can identify an integral lift in the place where the squares represent the Wu characteristic. In the context given above, the integral lifts of the Wu characteristic correspond to the torsion Pontrjagin characteristic.

Now we illustrate some examples and propositions in low degree.

1. Degree 2: for normal bundle : $v_2 = sw_2 + sw_1^2$. If sw_2 = 0, indicating a Pin+ structure, the torsion Pontrjagin characteristic P_2 provides a lift to the Wu (2) structure. We could also find $sw_2 = \rho_2(c_1)$, where $c_1$the first Chern characteristic.

2. Degree Six: $v_6 = sw_2 sw_4 + sw_3^2$. If sw_4 = 0, indicating an orientable case (e.g. membrane structure), reduces to the square term sw_3^2 . The integral lift of the Wu characteristic in this situation is the torsion Pontrjagin characteristic P_6 . We cannot
sw_2 = 0 as then sw_3, it would also be zero.

3. Degree ten: Here v_{10} will involve v_2. As a result, we are unable to isolate a square word. However, it will be provided by the torsion Pontrjagin characteristic P_{10} .

For instance, higher-order Stiefel-Whitney characteristics can be observed in an oriented 11-dimensional manifold $Y^{11}$where $sw_{11}(Y^{11}) = sw_{10}(Y^{11}) = sw_9(Y^{11}) = 0$. (Hisham, S. 2011). This excludes all phrases involving sw_{10} and sw_9 in v_{10}.

We can now describe Steenrod squares in the geometry domain.

First, if X is a manifold with dimension d, one may generate characteristics in

$Hom^n(X)$ by proper function $f: V \to X$ here V is a dimension d manifold, by intersection theory we can count n-cycle intersection points, as the pushforward $f^*(1)$ where 1 is the unit class in $Hom^0(V)$, as the Thom characteristic, or applying the basic characteristic in locally finite homology and duality. Using the last method, let's say that f has a normal bundle v since it is an immersion. Then $Stq^i(x)=f^*(sw^i(v))$ if $x = f^*(1) \in Hom^n(X)$. In essence, this is the Wu formula.

That is, if sub-manifolds contain cohomology properties, such as cup product approximating intersection data, Steenrod squares are normal bundle data.

The Steenrod squares $Stq^i: Hom^n(-;F_2) \to Hom^n + i(-;F_2)$ are basic cohomological functions. They indicate a map between the Eilenberg–MacLane spaces $K(F_2;n) \to K(F_2;n+i)$by the Yoneda lemma. By the Dold–Kan correspondence, this map should be expressible as a chain map

$$\widehat{Stq^i} : F_2[-n] \to F_2[-(n+i)].$$

We write sometimes $F_2 = Z_2Z$ for the field with two elemens.

Using operations and algebra of operations to explain some computations of the Steenrod operations on term of cohomology of a Hopf algebra over Z_p.

Let Hom* be the cohomology of the Hopf algebra A and investigate the Steenrod operations. Our goal is to apply these operations to E_2 (the bundle space) in a few fascinating particular circumstances. (William, M. 1973) (327-336).

We now discuss "Steenrod squares Stq_V^i, which are vertical, and compute Steenrod squares Stq_D^i, which are chain maps."

$$Stq_V^i : E_2^{p,q} \to E_2^{p,q+i}, \quad 0 \le i \le q$$
$$Stq_D^i : E_2^{p,q} \to E_2^{p+i,2q}, \quad 0 \le i \le p$$

Where $Stq_V^i = Stq^i$, $Stq_D^i = Stq^{i+q}$

In our initial application, we consider an extension of cocommutative Hopf algebras:

A → C → S. Let M be a commutative left C-algebra and N be a commutative left

S-algebra. Drawing parallels with group theory, we show that the action of B on $Ext_A(M,N)$ can be expressed directly as B's action on A. This involves Steenrod operations on E_2.

Then $Stq_V^i : Ext_B^p\left(Z_2, Ext_A^q(M.N)\right) \to Ext_B^p\left(Z_2, Ext_A^{q+i}(M,N)\right)$,

$$Stq_D^i : Ext_B^p\left(Z_2, Ext_A^q(M.N)\right) \to Ext_B^{p+i}(Z_2, Ext_A^{2q}(M,N)),$$

Then Stq_V^i acts on the cohomology of the Hopf algebra A, while Stq_D^i acts on the cohomology of B. (William, M. 1973) (327-336).

In the second scenario, let us consider a topological group G and a fundamental G-bundle E. The spectral sequence converges to $Hom^*(E/G)$ (with coefficients in Z_2), and the Steenrod operations on E appear as:

$$Stq_V^i : Ext_{Hom^*(G)}^{p,q}(Hom^*(E), Z_2) \to Ext_{Hom^*(G)}^{p,q+i}(Hom^*(E), Z_2),$$

$$Stq_D^i : Ext_{Hom^*(G)}^{p,q}(Hom^*(E), Z_2) \to Ext_{Hom^*(G)}^{p+i,2q}(Hom^*(E), Z_2)$$

We demonstrate that Stq_V^i determines the cohomology of the Hopf algebra on $Hom^*(G)$ *and* $Hom^*(E)$. Additionally, Stq_D^i defines operations on the cohomology of the Hopf algebra $Hom^*(G)$.

To satisfy the Cartan formula and Adem relations of vertical and diagonal squares, we use the Serre spectral sequence in our final application to ensure the Serre spectral sequence. To satisfy a fascinating commutation relation of vertical and diagonal squares, we find that vertical and diagonal squares satisfy: see sec6 (William,M. 1973) (327-336) .

$$. Stq_V^i \; Stq_D^j \to Stq_D^J \; Stq_V^{i/2}$$

Where $Stq_V^{i/2} = 0 \; if \; i \; is \; odd$

Conclusion:

Our study shows a deep composition of Steenrod square operations satisfies the following relations and calculate that the Steenrod squares with some applications

$Stq^i: Hom^n(X; F_2) \to Hom^{n+i}(X; F_2)$ are characterized by the following 5 axioms:

Our study demonstrates a deep composition of Steenrod square operations that meets the following relations and calculates that the Steenrod squares $Stq^i: Hom^n(X; F_2) \to Hom^{n+i}(X; F_2)$, with some applications are defined by the following 5 axioms:

1. Naturality: $Stq^i: Hom^n(X; F_2) \to Hom^{n+i}(X; F_2)$ is an additive homomorphism and is natural with respect to any $f: X \to Y$, so $f^*\left(Stq^i(x)\right) = Stq^i(f^*(x))$.

2. Stq^0 is the identity homomorphism.

3. $Stq^i(x) = x \smile x$ for $x \epsilon Hom^i(X; F_2)$.

4. If $i > \deg(x)$ then $Stq^i(x) = 0$

5. Cartan Formula: $Stq^i(x \smile y) = \sum_{n+m=i} Stq^n(x) \smile Stq^m(y)$

6. Additionally, the Steenrod squares have the following characteristics:

- Stq^1 is the Bockstein homomorphism β of the exact sequence $0 \to Z_2 \to Z_4 \to Z_2 \to 0$

- In cohomology, Stq^i commutes with the relating morphism of the long exact sequence. Specifically, it commutes in terms of suspension $Hom^K(X; F_2) \rightarrow Hom^{K+1}(\sum X\,; F_2)$.

- They satisfy the Adem relations, which are explained below. Similarly, the reduced $p-th$ powers are characterized by the following axioms for p > 2.

1. Naturality: $P^n\colon Hom^m(X; F_2) \rightarrow Hom^{2n(p+1)+m}(X; F_2)$ is an additive homomorphism and natural.
2. P^0 is the identity homomorphism.
3. P^n is the cup $p-th$ power of degree $2n$.
4. If $2n > \deg(x)$ then $P^n(x) = 0$
5. Cartan's Formula: $P^n(x \smile y) = \sum_{i+j=n} P^i(x) \smile P^j\,(y)$

REFERENCES:

[1] - Cohen,R. L. (August, 1998). The Topology of Fiber Bundles Lecture Notes. Dept of Mathematics Stanford University.

[2]-Elhamdadi, M. [2003]. On the Steenrod operations in cyclic cohomology, University of South Florida,Tampa, FL 33620, USA , South Valley University, Aswan, Egypt.

[3]- Giansiracusa, J and Deviate. (June 9, 2003). Stiefel-Whitney Characteristic.

[4]- Hisham, S. (21sept2011). Twisted topological structures related to M-branes II:Twisted Wu and Wu^c structures, University of Pittsburgh .Pittsburgh, PA 15260 Pennsylvania, USA.

[5] - Husemoller, D. (1994). Fibre Bundles. Third Edition. Editorial Board. (J.H. Ewing F.W. Gehring P.R. Halmos.), Berlin.

[6]- Husemoller, D. et al, (2008). Basic Bundle Theory and K-Cohomology Invariants, Lect.Notes Phys.726 (Springer, Berlin Heidelberg), DOI 10.1007/ 978-3-540-74956-1.

[7] – Janis, L. (2014). Characterizations of the Chern characteristic classes, Waterloo, Ontario, Canada.

[8] - Marathe, K. (2010). Topics in Physical Mathematics, DOI 10.1007/978-1-84882-939-8 1, Springer-Verlag London Limited.

[9]- Milnor, J.W. (1981); lectures on characteristic classes. University of Tokyo Press, Japan.

[10]- Nakahara, M. (2002). Geometry Topology and physics (second edition) Kinki University, Osaka, Japan.

[11]-William,M. singer, (janury1973). Steenrod squares in spectral sequences. II. Fordham university. New York, USA. (327-336).