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FULL PAPER

Review on Topological Cone Metric Spaces and Their Metrizability

Prepared by

Mohammed Rabih
Department of Mathematic
College of science
Qassim university
Buraydah 51452, Saudi Arabia
M.Fadlallah@qu.edu.sa,
mohammednoor81@gmail.com

Abstract

This survey study aims to compile the most recent works on cone metric spaces in a single location. This list may be beneficial for emerging scholars engaged in the domain of functional and nonlinear analysis. The Metrizability and topological characteristics of cone metric spaces are the primary focus of this investigation. This study demonstrates the Metrizability of cone metric spaces and generalizes some topological ideas and definitions to these spaces by using mathematical operators that are defined on ordered Banach spaces.

Keywords: Cone Metric Spaces, Topological Metrizability, Ordered Banach paces , Topological Structures, Fixed Point theorems

1. Introduction

Several fixed point theorems for contractive mappings were proved inside the freshly introduced notion of cone metric space here we refer to it as (CMS). In order to establish a CMS, the researchers exchange the real numbers $\mathbb{R}$ with an ordered Banach space E .

A series of articles about CMSs then started to appear. While some of those publications concentrated on the structure of those spaces, others addressed fixed point theorems (see [4], [6], [11], [18], [21], [33], [35], [36], [37], and [40]).

The interest in these spaces surged following the 2007 publication by L. G. Huang and X. Zhang [23], who redefined this category as CMSs. Their methodology involved utilizing the cone's inside points that delineated the space's partial order. The original study [23] also assumed the normality of the cone; nevertheless, subsequent conclusions were derived for non-normal cones, albeit with more complex proofs.

CMSs are currently being employed to investigate convergent sequences and Cauchy sequences, normed spaces, complete spaces, and metric spaces. These spaces are classified as topological, first countable, Hausdorff, and T_4 spaces, as evidenced by the resolution of topological inquiries in CMSs.

"Is a CMSs genuine generalisations of metric spaces, or are they equivalent?" will be a key question. Several writers have lately studied this topic, establishing the associated metric using a range of techniques and proving that CMSs are metrisable.

Each theorem in metric spaces is intrinsically holds in a CMS as we consider a CMS to be a legitimate metric space. The idea that CMSs are not a true generalisation of metric spaces is supported by this. We shall maintain the possibility of generalisations in the interim.

This paper is divided into five chapters. Chapter 1, serves as an introduction. Chapter 2, provides definitions of various functional spaces. In Chapter 3, we introduce the concept of CMSs. Chapter 4 will focus on the fundamental concepts of the Metrizable of metric cone spaces. The measurable nature of metric cone spaces (topological vectors) has been demonstrated in measurable prior investigations (e.g., [30], [46], [47], [49]). Nevertheless, which does not imply that all results of fixed-point in metric cone spaces are reducible to their standard metric counterparts, as minimum a few of them continue to be contingent upon the specific cone used. Consequently, research into these areas has persisted, and an increasing number of papers have been published. The majority of these papers are referenced in the present paper (although it is possible that we overlooked some; this was not the intention). Regarding recent research on the Metrizable of metric cone spaces (see, [14]), In Chapter 5, certain topological notions and definitions are extended to metric cone spaces in order to facilitate a more comprehensive examination of this subject (refer to References[5], [13], [22], [34], [38], [50]). The distance between two nonempty subsets, finite subsets, progressively closed subsets, normal space, T_4 space, continuous mappings, and sequentially continuous maps are among the topics that are discussed.

It was also shown that CMSs are T_4 spaces, Hausdorff spaces, first countable spaces, and topological spaces.

The Metrizable and topological characteristics of CMSs are the primary focus of this review paper, which sets it apart from its antecedents.

2. Preliminaries

The definitions of specific functional spaces are included in this chapter.

Definition 2.1. When X a real vector space with a map $\|\cdot\|: X \rightarrow \mathbb{R}$, referred known as the norm, constitute a real normed space in which:

- i) $\|x\| \geq 0$, for any element $x \in X$ and then $\|x\| = 0 \Leftrightarrow x = 0$.
- ii) $\|\alpha x\| = |\alpha| \|x\|$, for any element $x \in X$, α is a real value
- iii) $\|x + y\| \leq \|x\| + \|y\|$ for every x and y in X

Definition 2.2. Suppose we have a normed space that is complete, in this case it is called a Banach space.

Definition 2.3. Consider E as a real Banach space equipped with a norm $\|\cdot\|$, Hereafter we refer to it by the abbreviation RBSp

A cone is a not empty convex closed subset $K \subset E$ that satisfy all of the following conditions:

- i) $K \neq \{0\}$.
- ii) $\xi, \zeta \geq 0, \xi, \zeta \in \mathbb{R}$ and $x, y \in K$ implying that $\xi x + \zeta y \in K$.
- iii) $x \in K$ and $-x \in K$ implying that $x = 0$.

The cone $K \subset E$ can partially order the space E by the following condition: $x \leq y \Leftrightarrow y - x \in K$. We designate the interior of K as K° . We denote by $x \ll y$ (x is behind y) if $y - x \in K^\circ$. Additionally, the condition x

Definition 2.4. In $(E, \|\cdot\|)$, a cone K is referred to as

i. Normal: If we find $N > 0$ where N is a constant and:

$$0 \leq x \leq y \text{ Which means } \|x\| \leq N\|y\|.$$

The smallest possible integer greater than zero is K normal constant. If N is always less than 1, it will be evident that there are no cones.

ii. Regular: If any sequence is increasing and bounded above that is convergent.

To put it another way, if $\{x_k\}_{k \geq 1}$ be any sequence where $x_1 \leq x_2 \leq \dots \leq y$ for certain $y \in E$, then we found $x \in E$ for which $\lim_{k \rightarrow \infty} \|x_k - x\| = 0$. In other words, the cone K is called regular only if each decreasing sequence circumscribed below is convergent.

There are normal cones that are not regular, as evidenced by [39] and [45]. Cones that are regular are normal.

iii- Minihedral: Any set that sub of E is bounded from above has a supremum value if $\sup\{x, y\}$ one can find, $\forall x, y \in E$, and hence the space is said to be it is strongly minihedral.

iv. Solid: If K° is not empty.

Proposition 2.5.

- i- Normal cone with normal constant $N < 1$ do not exist.
- ii- All normal cone that are strongly minihedral are regular.
- iii- Normal cone that are not regular can exist.

3. Cone metric spaces (CMSs)

when K is a cone in E when E is an RBSp, and the partial ordering is provided in according to K .

Definition 3.1. Assume that $X \neq \emptyset$ and a mapping D from $X \times X$ to E satisfying the following properties

- (i) $0 \leq D(x, y)$, for any two elements $x, y \in X$ together with $D(x, y) = 0$ holds if and only if $x = y$.
- (ii) $D(x, y) = D(y, x)$.
- (iii) $D(x, y) \leq D(x, z) + D(z, y), \forall x, y, z \in X$.

The pair $(X, \mathcal{D})$ is thereafter known as a cone metric space (CMS), and $\mathcal{D}$ is referred to as a cone metric on X .

Definition 3.2. Assume that we have a cone metric $(X, \mathcal{D})$, $x \in X$ and sequence $\{x_k\}$ in X , then:
(i) When there is N where, in the case that $k > N$, $(x_k, x) \ll c$, $\{x_k\}$ is regarded as convergent to $x \in X$ for every element $c \in E$ satisfying $c \gg 0$.

When in the case that is N satisfying for any element $c \in E$ with $c \gg 0$, $\mathcal{D}(x_k, x) \geq c$, then $\{x_k\}$ is defined to be convergent to $x \in X$.

(ii) If for each $c \in E$, $c \gg 0$, there exists N satisfying for any two element $k, h > N$, $\mathcal{D}(x_k, x_h) \ll c$, then $\{x_k\}$ is a Cauchy sequence in X .

(iii) The CMS $(X, \mathcal{D})$ is said to be complete provided that every Cauchy sequence with respect to the cone metric $\mathcal{D}$ converges to a point in

Definition 3.3. Cone Normed Spaces, Suppose X is a real vector space. and the mapping $\|\cdot\|_p: X \rightarrow E$ satisfying:

(i) $\|x\|_p \geq 0$, $\forall x \in X$ and $\|x\|_p = 0 \Leftrightarrow x = 0$.

(ii) $\|\alpha x\|_p = |\alpha| \|x\|_p$, $\forall x \in X$ and $\alpha \in \mathbb{R}$.

(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$, $\forall x, y \in X$.

Then, $(\|\cdot\|_p, X)$ is referred to as a cone normed space, and $\|\cdot\|_p$ is referred to as a cone norm on X .

By replacing $E = \mathbb{R}$, with $K = [0, \infty]$, it is simple to demonstrate that any normed space is a cone normed space.

Proposition 3.4. All cone-normed spaces are topological spaces. The topology is really provided by:

$T_K = \{\mathcal{U} \subset X: \text{for all } x \in X, \text{there exist } c \gg 0 \text{ Given that } B(x, c) \subset \mathcal{U}\}$ when;

$B(x, c) = \{y \in X: \|(x - y)\|_p \ll c\}$.

Definition 3.5. If there can be found $\in T_K$, $x \in \mathcal{U}$ for which $T(\mathcal{U}) \subset \mathcal{W}$ to any $\mathcal{W} \in T_K$ containing $T(x)$, then a map T from $(X, \mathcal{D})$ to $(X, \mathcal{D})$ is said to be continuous for the element $x \in X$. T is called continuous if it is a continuous map at each $x \in X$.

Definition 3.6. If the set $\{\|x\|_p: x \in A\}$ has an upper bound, and we have $(X, \|\cdot\|_p)$ is a cone normed space, then a subset A of X is a bounded cone.

4. metrizable of cone metric spaces

A topological space that is homeomorphic to a metric space is referred to as a metric space. In other words, if there exists a metric $d: X \times X \rightarrow [0, \infty]$ such that the topology induced by d coincides with T , then the topological space (X, T) is metrisable.

Urysohn's theorem was among the first known metrization theorems. It is claimed that all Hausdorff second-countable regular spaces are metrisable. Numerous other metrization theorems follow Urysohn's theorem as simple corollaries. For instance, if and only if a compact Hausdorff space is second-countable, then it is metrizable. Any ordinary conic space is a topological space, as we explained in the last chapter; it just retains the analogous metric d .

Note: It is true that any CMS is metrisable.

Using different techniques, several writers defined the equivalent metric and proved that the cone metric spaces (CMSs) are metrizable. We will look at some of such methods in this chapter and end with a comment on the subject.

First, M. Khani and M. Pourmadian proved that CMS is metrizable [30].

The metric d that represents a CMS is defined by the following theorem.

Theorem 4.1. Assume that $c < 1$ is in $\mathbb{R}^+$, $\alpha \in K^\circ$, moreover, the quadruple (X, E, K, D) defines a CMS. Then, there is a metric $d: X \times X \rightarrow \mathbb{R}^+$ that causes X to have the same topology as that induced by cone metric topology that $\mathcal{D}$ causes. Furthermore, if and only if a sequence $\{x_k\}$ is Cauchy in (X, d) , it is also Cauchy in $(X, E, K, \mathcal{D})$. In particular, if and only if $(X, \mathcal{D})$ is complete, then $(X, E, K, \mathcal{D})$ is complete CMS.

Their definition was:

$$A(x, y) = \begin{cases} \mathcal{D}^{\min\{N:d(x,y) < d^N\alpha\}} \\ 0, & \mathcal{D}(x, y) = 0 \end{cases}, \mathcal{D}(x, y) \neq 0$$

$$\text{and } d(x, y) = \inf\left\{\sum_{i=1}^{k-1} A(x_i, x_{i+1}): x_1 = x, \dots, x_k = y\right\}$$

CMSs can be partially compared to the well-known metric spaces, despite their challenging definition.

S. Al-Dwaik and A.A. Hakawati attempted to tackle the problem of metrizable in the context of the best approximation in [2]. Readers are directed to [42] for the traditional setting of best approximation theory.

Also, M. Asadi, and H. Soleimani, S.M. Vaezpour in [27] endeavoured to establish the metrizable by establishing that every cone space is metrisable and that the corresponding metric fulfils the same contractive properties as the cone metric, thereby addressing the significant question "Are CMSs a genuine generalisation of metric spaces?"

When $d: X \times X \rightarrow \mathbb{R}^+$ is the metric that corresponds to $\mathcal{D}$ and $\mathcal{D}: X \times X \rightarrow E$ is any cone metric, they defined $d(x, y) = \inf\{\|\mu\|: \mathcal{D}(x, y) \leq \mu\}$.

In addition, the same authors in [28] demonstrate that CMSs are metrisable through an appropriate renorming of the Banach space. Therefore, a constant $N = 1$ can be employed to convert any cone K into a regular cone.

The following was their renorming. $\|\cdot\|: E \rightarrow [0, \infty)$ is defined as follows: $\forall x \in E, \|x\| = \inf\{\|\mu\|: x \leq \mu\} + \inf\{\|w\|: w \leq x\} + \|x\|$,

Therefore, the metric described by $d(x, y) = \|\mathcal{D}(x, y)\|$ is identical to any cone metric $\mathcal{D}: X \times X \rightarrow E$. Each cone metric that is introduced on a Banach space is therefore actually metrically equivalent to a metric.

The results of M. Asadi et al. in [28] were disputed by Z. Kadelburg and S. Radenovich in [48]. This is due to a mistake in the final section of the argument, which shows that P is monotone (normal cone with normal constant $N = 1$) in the new norm, which leads to an inaccurate conclusion. Such an assumption would imply that every cone in Banach spaces is normal, thereby is obviously untrue.

Furthermore, a number of authors in [20] disputed the results of M. Asadi et al. Their writings are also titled "On the non-metrizability of CMSs." The following counterexample, supplied by the authors, shows that the main theorem in [28] is invalid.

Remark 4.2. [17], transforms every strongly minihedral normal cone into a normal cone with $N = 1$ by adding a new norm to our Banach space. As a result, all CMSs are metric spaces, and all theorems in metric spaces are inherently applicable to CMSs, thus follows

Lemma 4.3. For $x, y \in E$, we have the following if K is a highly minihedral cone in an RBS E :

$$(1) \inf\{\omega: \omega \geq x + y\} = \inf\{\mu: \mu \geq x\} + \inf\{w: w \geq y\}$$

$$(2) \sup\{\omega: \omega \leq x + y\} = \sup\{\mu: \mu \leq x\} + \sup\{w: w \leq y\}$$

Proof:

Assume that $\mathcal{U} = \{\omega: \omega \geq x + y\}$, $\mathcal{U}_1 = \{\mu: \mu \geq x\}$, $\mathcal{W}_1 = \{w: w \geq y\}$

Let σ , σ_1 and σ_2 be the $\inf(\mathcal{U})$, $\inf(\mathcal{U}_1)$ and $\inf(\mathcal{W}_1)$, correspondingly.
subsequently, if $\mu \in \mathcal{U}$ then $\mu \geq x + y$, so $\mu - y \geq x$ and consequently $\mu - y \in \mathcal{U}_1$
So $\sigma_1 \leq \mu - y \Rightarrow \mu - \sigma_1 \geq y$ as well as $\mu - \sigma_1 \in \mathcal{W}_1$.
So $\sigma_2 \leq \mu - \sigma_1$ and so $\sigma_1 + \sigma_2 \leq \mu$ (This is accurate. $\forall \mu \in \mathcal{U}$). So

$$\sigma_1 + \sigma_2 \leq \sigma \# (1)$$

Regarding the other inequality, take notice that $\sigma_1 + \sigma_2 \geq x + y$ by definition, meaning that $\sigma_1 + \sigma_2 \in \mathcal{U}$, and therefore

$$\sigma \leq \sigma_1 + \sigma_2 \# (2)$$

by (1) and (2) We find that : $\sigma = \sigma_1 + \sigma_2$ namely $\inf\{\omega: \omega \geq x + y\} = \inf\{\mu: \mu \geq x\} + \inf\{w: w \geq y\}$

In a similar way, $\sup\{\omega: \omega \leq x + y\} = \sup\{\mu: \mu \leq x\} + \sup\{w: w \leq y\}$.

Lemma 4.4. Considering $0 \leq x \leq y$ it follows that:

$$(1) \|\sup\{\mu: \mu \leq x\}\| \leq \|\sup\{\mu': \mu' \leq y\}\|$$

$$(2) \|\inf\{w: x \leq w\}\| \leq \|\inf\{w': y \leq w'\}\|$$

Proof: For (1) Assume that $\mathcal{U} = \{\mu: \mu \leq x\}$, $\mathcal{U}' = \{\mu': \mu' \leq y\}$

Given that $0 \leq x \leq y$ this implies the expression $x \in \mathcal{U}' \Rightarrow x \leq \sup(\mathcal{U}')$ therefore

$\forall \mu \in \mathcal{U}$ We hold $\mu \leq \sup(\mathcal{U}')$ therefore $\sup(\mathcal{U}) \leq \sup(\mathcal{U}')$ it follows that,

$$\|\sup\{\mu: \mu \leq x\}\| \leq \|\sup\{\mu': \mu' \leq y\}\|$$

Similarly, one can show: $\|\inf\{w: x \leq w\}\| \leq \|\inf\{w': y \leq w'\}\|$.

Theorem 4.5. Assume $(E, \|\cdot\|)$ is an RBSp with a normal cone K that is highly minihedral. Thus, With the normal constant $N = 1$, a cone K is a normal, , and one can define a norm $[\cdot]$ on E with regard to it.

Proof : State $[\cdot]: E \rightarrow [0, \infty)$ as follows: $[x] = \|\inf\{\mu: x \leq \mu\}\| + \|\sup\{w: w \leq x\}\|$, $\forall x \in E$

It is evident that , if $x = 0$ implies $[x] = 0$, for all $x \in E$ If $[x] = 0$ then $\exists \mu_k, w_k \in E$ satisfying: $w_k \leq x \leq \mu_k$ when $\mu_k \rightarrow 0, w_k \rightarrow 0$ with $k \rightarrow \infty$. We obtain $x = 0$ since K is a regular cone.

Accordingly, $[x] = 0$ if only $x = 0$.

$$\begin{aligned} \text{At this point } [-x] &= \|\inf\{\mu: -x \leq \mu\}\| + \|\sup\{w: w \leq -x\}\| \\ &= \|\sup\{-\mu: -\mu \leq -x\}\| + \|\inf\{-w: -w \leq -x\}\| \\ &= \|\sup\{-\mu: -\mu \leq x\}\| + \|\inf\{-w: x \leq -w\}\| \\ &= \|\sup\{\mu: \mu \leq x\}\| + \|\inf\{w: x \leq w\}\| = [x] \end{aligned}$$

If Λ is greater than zero

$$\begin{aligned} [\Lambda x] &= \|\inf\{\mu: \Lambda x \leq \mu\}\| + \|\sup\{w: w \leq \Lambda x\}\| \\ &= \left\| \inf\left\{\Lambda \left(\frac{1}{\Lambda} \mu\right): x \leq \frac{1}{\Lambda} \mu\right\} \right\| + \left\| \sup\left\{\Lambda \left(\frac{1}{\Lambda} w\right): \frac{1}{\Lambda} w \leq x\right\} \right\| \\ &= \Lambda \left\| \inf\left\{\frac{1}{\Lambda} \mu: x \leq \frac{1}{\Lambda} \mu\right\} \right\| + \Lambda \left\| \sup\left\{\frac{1}{\Lambda} w: \frac{1}{\Lambda} w \leq x\right\} \right\| = \Lambda [x] \end{aligned}$$

Consequently, $[\Lambda x] = |\Lambda| [x]$, for any $x \in E$ with a real element Λ

The triangle inequality is now demonstrated using lemma 3.4.

$$\begin{aligned} [x + y] &= \|\inf\{\mu: x + y \leq \mu\}\| + \|\sup\{w: w \leq x + y\}\| \\ &= \|\inf\{\mu: x \leq \mu\} + \inf\{\mu: \mu \leq y\}\| + \|\sup\{w: w \leq x\} + \sup\{w: w \leq y\}\| \\ &\leq \|\inf\{\mu: x \leq \mu\}\| + \|\inf\{\mu: \mu \leq y\}\| + \|\sup\{w: w \leq x\}\| + \\ &\|\sup\{w: w \leq y\}\| \\ &= \|\inf\{\mu: x \leq \mu\}\| + \|\sup\{w: w \leq x\}\| + \|\inf\{\mu: \mu \leq y\}\| + \\ &\|\sup\{w: w \leq y\}\| \\ &= [x] + [y]. \end{aligned}$$

Thus, $[\cdot]$ induces a norm on E since $[x + y] \leq [x] + [y]$.

Finally, with respect to this norm $[\bullet]$, by lemma.4.4, K is a normal cone with normal constant $N = 1$.

We believe that every theorem in metric spaces is necessarily valid in a CMS since a cone CMS is a real metric space. CMSs are not a true generalisation of metric spaces, as this further supports.

Table 1. A Comparative Study of Metrizable Techniques in cone metric spaces

Author(s)	Year	Cone Assumptions	Main Technique	Result	Limitations
Huang & Zhang	2007	Normal, solid cone	Approach from the inside	Metrizability	Needs a lot of normalcy
Khani & Pourmadian	2010	Normal cone	Path-based metric building	Topological similarity	Building something complicated
Asadi et al.	2012	Random cone	Renorming a Banach space	Equivalence in the metric system	Criticised later
Hakawati & Sarries	2016	Ba-nach space that is ordered	Explicit metric through order	Clear decrease	Depends on how the order is set up
Kadelburg & Radenović	2013	—	Analysis of counterexamples	Disagreement	Demonstrates the constraints of enorming.
Recent works (2020–2024)	—	Different	Hybrid or application-based	Partial equivalence	Results that depend on the cone

5. Topological cone metric spaces

This chapter addressed topological considerations in CMSs and illustrated that examples of CMSs encompass topological spaces, first countable spaces, Hausdorff spaces, and T_4 spaces. Additional fundamental topological concepts and definitions are likewise extended within CMSs. The following essential lemmas must be established as a priority:

Lemma 5.1. Suppose that $(X, \mathcal{D})$ constitutes a CMS. Consequently, for any $c > 0$ such that $c \in E$, one can find a $\delta > 0$ satisfying $(c - x) \in K^\circ$ (i.e. $x \ll c$) provided that $\|x\| < \delta$ and $x \in E$.

Lemma 5.2. Let us assume that the metric space $(X, \mathcal{D})$ is a cone. one can find $c \gg 0, c \in E$ for which $c \not\geq c_1$ and $c \not\geq c_1$ if $c_1 \gg 0$ and $c_2 \gg 0, c_1, c_2 \in E$.

Using Lemma 5.1, we may determine that $\delta > 0$ in such a way that $\|x\| < \delta$, indicating that $x \not\geq c_2$, since $c \gg 0$. Make sure that $\frac{1}{k_0} < \frac{\delta}{\|c_1\|}$ by selecting k_0 . Let $c = \frac{c_1}{k_0}$. Therefore, $c \not\geq c_2$ since $\|c\| = \left\| \frac{c_1}{k_0} \right\| = \frac{\|c_1\|}{k_0} < \delta$. However, it is also evident that $c \not\geq c_1$ and $c \gg 0$.

Theorem 5.3. All of the CMSs $(X, \mathcal{D})$ are topological spaces.

Proof: $\forall c \gg 0, c \in E$, take $B(x, c) = \{y \in X : \mathcal{D}(x, y) \ll c\}$ and $\beta = \{B(x, c) : x \in X, c \gg 0\}$ so, on X , $T_c = \{\mathcal{U} \subset X : \forall x \in \mathcal{U}, \exists B \in \beta, x \in B \subset \mathcal{U}\}$ is a topology. Actually, we have:

(**M₁**) $\emptyset, X \in T_c$.

(**M₂**) Take $\mathcal{U}, \mathcal{W} \in T_c$ and take $x \in \mathcal{U} \cap \mathcal{W}$. Then, $x \in \mathcal{U}$ and $x \in \mathcal{W}$, find $c_1 \gg 0$, $c_2 \gg 0$ satisfying $x \in B(x, c_1) \subset \mathcal{U}$ and $x \in B(x, c_2) \subset \mathcal{W}$, with lemma 5.2, find $c \gg 0$ in such a way that $c \ll c_1$ and $c \ll c_2$. It follows immediately that $x \in B(x, c) \subset B(x, c_1) \cap B(x, c_2) \subset \mathcal{U} \cap \mathcal{W}$. Therefore, $\mathcal{U} \cap \mathcal{W} \in T_c$.

(**M₃**) Take $\mathcal{U}_\alpha \in T_c$ for each element $\alpha \in \Gamma$, and let $x \in \bigcup_{\alpha \in \Gamma} \mathcal{U}_\alpha$. Then $\exists \alpha_0 \in \Gamma$ satisfying that $x \in \mathcal{U}_{\alpha_0}$. Therefore, find $c \gg 0$ for which $x \in B(x, c) \subset \mathcal{U}_{\alpha_0} \subset \bigcup_{\alpha \in \Gamma} \mathcal{U}_\alpha$. That is, $\bigcup_{\alpha \in \Gamma} \mathcal{U}_\alpha \in T_c$.

Theorem 5.4. A Hausdorff space is any CMS $(X, \mathcal{D})$.

Proof: Assuming that $x, y \in X$ such that $(x \neq y)$ are two points in X , $\mathcal{D}(x, y) = c > 0$, resulting in $\left[B\left(x, \frac{c}{2}\right) \cap B\left(y, \frac{c}{2}\right)\right] \in T_c$ and $B\left(x, \frac{c}{2}\right) \cap B\left(y, \frac{c}{2}\right) = \emptyset$.

Definition 5.5. If we assume that $\mathcal{U} \neq \emptyset$ and $\mathcal{W} \neq \emptyset$ are pair of subsets of a CMS $(X, \mathcal{D})$, In this case $d(\mathcal{U}, \mathcal{W}) = \inf\{\mathcal{D}(\mu, w) : \mu \in \mathcal{U}, w \in \mathcal{W}\}$ is the distance between $\mathcal{U}$ and $\mathcal{W}$. We write $d(\mu, \mathcal{W})$ for $d(\mathcal{U}, \mathcal{W})$. if $\mathcal{U} = \{\mu\}$.

Theorem 5.6. All of the CMSs $(X, \mathcal{D})$ are first countable..

Proof: Exactly $c \leq 0$ when $c \in E$ and assume that $q \in X$. $\beta_q = \left\{B\left(q, \frac{c}{k}\right) : k \in \mathbb{N}\right\}$ is a local basis at q , as can be seen. Let $q \in \mathcal{U}$ and $\mathcal{U}$ be open.

Using lemma 5.1, find k_0 such that $\frac{c}{k_0} \ll c_1$ and $c_1 \gg 0$ in a manner that ensures

$q \in B(q, c_1) \subset \mathcal{U}$. Since $B\left(q, \frac{c}{k_0}\right) \subset B(q, c_1) \subset \mathcal{U}$, this follows.

Definition 5.7. Assume that the metric space $(X, \mathcal{D})$ is a cone. And $\mathcal{U}$ is a subset of X is said to be in the sequential sense closed if $x_k \in \mathcal{U}$ whenever $x_k \rightarrow x$.

Theorem 5.8. Assume $\mathcal{U} \subset X$, where $\mathcal{U} \neq \emptyset$, and that $(X, \mathcal{D})$ is a CMS. If only $\mathcal{D}(x, \mathcal{U}) = 0$, then $x \in \bar{\mathcal{U}}$.

Proof : Assume x is a member of $\bar{\mathcal{U}}$. Consider, $B\left(x, \frac{c}{k}\right) \cap \mathcal{U} \neq \emptyset$ for $c \gg 0$ and any element $k \in \mathbb{N}$. It follows that for any k in $\mathbb{N}$, There exists μ_k in $\mathcal{U}$ in a manner that ensures $0 \leq \mathcal{D}(x, \mathcal{U}) \leq \mathcal{D}(x, \mu_k) < \frac{c}{k}$, Therefore, for any k in $\mathbb{N}$, $\|\mathcal{D}(x, \mathcal{U})\| \leq N \frac{\|c\|}{k}$, Consequently, $\mathcal{D}(x, \mathcal{U}) = 0$.

On the other hand, we can choose $c \gg 0$ for which $B(x, c) \subset \mathcal{W}$ if $\mathcal{W} \in T_c$ is open in $(X, \mathcal{D})$ for which $x \in \mathcal{W}$. Find $\mu \in \mathcal{U}$ satisfying $\mathcal{D}(x, \mu) < c$, however, because $0 = \mathcal{D}(x, \mathcal{U}) < c$. In other words, $\mu \in \mathcal{U} \cap B(x, c) \subset \mathcal{U} \cap \mathcal{W}$.

Definition 5.9. Topological space X is a normal space if every pair of sets F and G closed, and $F \cap G = \emptyset$ have open, disjoint neighbourhoods $\mathcal{U}$ of F and $\mathcal{W}$ of G . It makes more logical that this condition allows F and G to be neighbourhoods.

Definition 5.10. A T_4 space is a normal T_1 space X , equivalently is the same as X being Hausdorff and normal.

Theorem 5.11. T_4 spaces are all CMSs $(X, \mathcal{D})$.

Proof: $(X, \mathcal{D})$ is a Hausdorff space, as was previously stated. Let X have two closed disjoint subsets, F and G . Then, define $\mathcal{U} = \{x \in X : \mathcal{D}(x, F) < \mathcal{D}(x, G)\}$. By the definitions of $\mathcal{U}$ and $\mathcal{W}$, we can see that $\mathcal{U} \cap \mathcal{W} = \emptyset$. This illustrates that X is normal. Additionally, $\mathcal{D}(\xi, F) = 0, \xi \notin G$, and noting that G is closed, $\mathcal{D}(\xi, G) > 0$ if $\xi \in F$. Theorem 5.8 states that if $\xi \in \mathcal{U}$, then $F \subset \mathcal{U}$ since $\mathcal{D}(\xi, F) < \mathcal{D}(\xi, G)$.

We will now be finished if we demonstrate that $\mathcal{U}$ and $\mathcal{W}$ are open. Consider $x_0 \in \mathcal{U}$, which implies that $c_1 = \mathcal{D}(x_0, F) < \mathcal{D}(x_0, G) = c_2$ to demonstrate that $\mathcal{U}$ is

open. We can define $c = \frac{1}{2}(c_2 - c_1)$ and take into consideration the fundamental open $B\left(x_0, \frac{c}{2}\right)$ since $c_2 - c_1 > 0$ [that is, $c_2 - c_1 \in K, c_2 \neq c_1$]. Given $x \in B\left(x_0, \frac{c}{2}\right)$, we can choose $\xi \in \mathcal{F}$ for which $\mathcal{D}(x_0, \xi) < c_1 + s$ for any $s \gg 0$ according to the definition of $\mathcal{D}(x_0, \mathcal{F})$. $\mathcal{D}(x, \mathcal{F}) \leq \mathcal{D}(x, \xi) \leq \mathcal{D}(x, x_0) + \mathcal{D}(x_0, \xi) < \frac{c}{2} + c_1 + s$, as a result. Consequently, $\mathcal{D}(x, \mathcal{F}) \leq \frac{c}{2} + c_1 = \frac{1}{4}(3c_1 + c_2)$. follows. Additionally, since $\mathcal{D}(x_0, \mathcal{G}) \leq \mathcal{D}(x_0, \zeta)$ and $\mathcal{D}(x, x_0) \leq \frac{c}{2}$, we obtain $\mathcal{D}(\zeta, x_0) \leq \mathcal{D}(\zeta, x) + \mathcal{D}(x, x_0)$ for $\zeta \in \mathcal{G}$. $\mathcal{D}(\zeta, x) + \frac{c}{2} > \mathcal{D}(x_0, \mathcal{G}) = c_2$ is one possible formula. $\mathcal{D}(\zeta, x) > c_2 - \frac{c}{2} = \frac{1}{4}(3c_2 + c_1)$, we can conclude. Next, we determine that $\mathcal{D}(x, \mathcal{F}) < \mathcal{D}(x, \mathcal{G})$ by seeing that $c_2 + 3c_1 < 3c_2 + c_1$. In other words, $\mathcal{U}$ is an open subset of X since $x \in \mathcal{U}$. The same logic demonstrates that $\mathcal{W}$ is likewise an open subset of X .

Table 2: A comparison of the topological properties induced by cone metrics and classical metrics.

Topological Property	Classical Metric Spaces	Cone Metric Spaces	Condition on the Cone
Hausdorff T_2	Always true	True	Cone has nonempty interior
First countable	Always true	True	Normal cone
Sequential	Equivalent to topological	Equivalent	Normal cone
Normal	Not automatic	Holds	Normal cone
T_4	Holds	Holds	Normal cone
Metrisable topology	Yes	Yes	Solid cone

Definition 5.12. If there can be found a $\mathcal{U} \in T_c$ containing x for which $T(\mathcal{U}) \subset \mathcal{W}$ for any $\mathcal{W} \in T_c$ containing $T(x)$, then the map $T: (X, \mathcal{D}) \rightarrow (X, \mathcal{D})$ is said to be continuous for $x \in X$. T is referred to as continuous if it is continuous at every $x \in X$.

Definition 5.13. Consider the metric space $(X, \mathcal{D})$ is a cone. Sequentially continuous maps are those for which $x_k \in X$ for which $x_k \rightarrow x$ leads to $T(x_k) \rightarrow T(x)$.

Theorem 5.14. Consider $T: (X, \mathcal{D}) \rightarrow (X, \mathcal{D})$ is a map and that $(X, \mathcal{D})$ is a CMS. If and only if T is in the sequential sense continuous, then T is continuous.

Proof: Since T is continuous at $x \in X$, if $x_k \rightarrow x$ and $c \gg 0$, we can choose $c_1 \gg 0$ satisfying $T(B(x, c_1)) \subset (B(T(x), c))$. Determine no such that $\mathcal{D}(x_k, x) < c_1, \forall k \geq k_0$, by using the convergence of x_k . $\mathcal{D}(T(x_k), T(x)) \ll c, \forall k \geq k_0$, nevertheless. The opposite is true since $(X, \mathcal{D})$ is a topological space that is first countable.

Definition 5.15. Let $c \gg 0, c \in E$ and $(X, \mathcal{D})$ be a CMS. For the subset $\mathcal{U} \subset X$, a finite subset $\mathcal{H} = \{c_1, c_2, \dots, c_k\}$ of X is referred to as a c -net if, for every $p \in \mathcal{U}$, there exists $c_i \in \mathcal{H}$ satisfying $\mathcal{D}(p, c_i) \ll c$.

Definition 5.16. Assume that the metric space $(X, \mathcal{D})$ is a cone. Given any $c \gg 0, c \in E, \mathcal{U}$ admits assembled into a finite union of sets $\mathcal{H}_i, i = 1, 2, \dots, n$ (i.e. $\mathcal{U} \subset \bigcup_{i=1}^n \mathcal{H}_i$), where $\delta(\mathcal{H}_i) \ll c$, then the subset $\mathcal{U}$ of $(X, \mathcal{D})$ is said to be entirely bounded.

Theorem 5.17. Assume that $\mathcal{U} \subset X$ and $(X, \mathcal{D})$ is a CMS. If and only if $\mathcal{U}$ has a c -net for any $c \gg 0, c \in E$, then $\mathcal{U}$ is totally bounded.

Proof: Suppose that $\mathcal{U}$ is completely limited, and allow $c \gg 0, c \in E$. Next, determine $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_k$ such that $\mathcal{U} \subset \bigcup_{i=1}^k \mathcal{H}_i, \delta(\mathcal{H}_i) \ll c$

Select an element c_i from each $\mathcal{H}_i$ such that $i = 1, 2, \dots, n$. Assume that $\mathcal{H} = \{c_1, c_2, \dots, c_k\}$. For $\mathcal{U}$, we demonstrate that $\mathcal{H}$ is a c -net.

Assume that p belongs to $\mathcal{U}$. In such case, $p \in \mathcal{H}_{i_0}$ since $c_{i_0}, i_0 = \{1, 2, \dots, k\}$. Given that $\delta(\mathcal{H}_i) \ll c$ and that both p and $\mathcal{G}_{i_0}$ are in $\mathcal{H}_i$, we can argue that $\mathcal{D}(p, c_i) \ll c$.

On the other hand, suppose $c \gg 0$ and $c \in E$. There exist a finite set $\mathcal{H} = \{c_1, c_2, \dots, c_k\}$ such that $\mathcal{D}(p, c_{i_0}) \ll c$ for every $p \in \mathcal{U}$, so that $c_{i_0} \in \mathcal{H}$. Let $i = 1, 2, \dots, k$, and $c_{i_0} = B(c_i, c) = \{x \in X : \mathcal{D}(x, c_i) \ll c\}$. Then it is evident that $\mathcal{U} \subset \bigcup_{i=1}^k \mathcal{H}_i$ and $\delta(\mathcal{H}_i) \ll c$.

Definition 5.18. Assume that the metric space $(X, \mathcal{D})$ is a cone. A Lebesgue element of a cover is an element $c \gg 0, c \in E$. $C = \{\mathcal{G}_i\} \subset T_c$, where $i = 1, 2, 3, \dots$. If any subset $\mathcal{W}$ of a subset $\mathcal{U}$ of $(X, \mathcal{D})$ has $\delta(\mathcal{W}) < c$, then there exists $\mathcal{G}_{i_0} \in C$ such that $\mathcal{W} \subset \mathcal{G}_{i_0}$.

Definition 5.19. Take $(X, \mathcal{D})$ as a CMS. If all cover of $\mathcal{U}$ by subsets from T_c contains a finite subcover that also covers T_c , then the subset T_c of $(X, \mathcal{D})$ is said to be compact.

Definition 5.20. take $\mathcal{U} \subset X$ be a CMS and $(X, \mathcal{D})$ denote a cone. $\mathcal{U}$ is said to be in the sequential sense compact if, for any sequence $\{x_k\}$ within $\mathcal{U}$, there's a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ for which $\{x_{k_i}\}$ converges to a point in $\mathcal{U}$.

Theorem 5.21. Assume that $\mathcal{U} \subset X$ and $(X, \mathcal{D})$ be a CMS. $\mathcal{U}$ is completely confined if it is sequentially compact.

Proof: Assume that $\mathcal{U}$ cannot have a c -net if we may choose a $c \gg 0$ for which $c \in E$. Therefore, For an arbitrary $x_1 \in \mathcal{U}$, there can be found $x_2 \in \mathcal{U}$ satisfying: $c - \mathcal{D}(x_1, x_2) \notin K^\circ$, and $\{x_1, x_2\}$ cannot be a c -net for $\mathcal{U}$. Consequently, there exists $x_3 \in \mathcal{U}$ for which $c - \mathcal{D}(x_1, x_3) \notin K^\circ$ and $c - \mathcal{D}(x_3, x_2) \notin K^\circ$. We create a sequence $x_k \in \mathcal{U}$ in a similar manner, so that $c - \mathcal{D}(x_k, x_h) \notin K^\circ, \forall k, h \in \mathbb{N}$. Therefore, $\{x_k\}$ cannot have a convergent subsequence, therefore any subsequence of $\{x_k\}$ cannot be Cauchy. $\mathcal{U}$ is not sequentially compact as a result.

Theorem 5.22. Assume that the metric space $(X, \mathcal{D})$ is a cone. If and only if $\mathcal{U}$ is in terms of sequences compact, then a subset $\mathcal{U} \subset X$ is compact.

Proof: For any $\mathcal{U}$ have an open cover $C = \{\mathcal{G}_i\}_{i \in I}$. $\mathcal{U}$ is sequentially compact, so given any subset $\mathcal{W} \subset \mathcal{U}$ satisfying $\delta(\mathcal{W}) < c$, there is $i_0 \in I$ with $\mathcal{W} \subset \mathcal{G}_{i_0}$ where $c \gg 0$ where $c \in E$. Considering that $\mathcal{U}$ is completely limited, $\mathcal{U} \subset \bigcup_{i=1}^k \mathcal{H}_i$, where $\delta(\mathcal{H}_i) \ll c$. Therefore, find $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k \in C$ such that $\mathcal{H}_i \subset \mathcal{G}_i$ for each $i = 1, 2, \dots, k$. In other words, $\mathcal{U}$ is compact since $\mathcal{U} \subset \bigcup_{i=1}^k \mathcal{H}_i \subset \bigcup_{i=1}^k \mathcal{G}_i$.

Nevertheless, it is noteworthy to observe that every CMS is a topological space.

Conclusion

This review has analysed the essential topological and metrizable characteristics of CMSs. We have demonstrated that CMSs inherently generate topological structures that are first countable, Hausdorff, and T_4 . Furthermore, numerous metrizable results establish that, given appropriate conditions on the cone, CMSs are topologically equivalent to traditional metric spaces.

The study further generalises fundamental concepts such as continuity, sequential continuity, total boundedness, and compactness within the framework of CMSs, illustrating significant parallels with classical metric theory. Although CMSs integrate structures of ordered Banach spaces, the findings suggest that, in most instances, they do not constitute a true topological generalisation of metric spaces. Nonetheless, their framework continues to be pertinent for particular applications, especially in fixed point theory, where the selection of the cone may continue to affect the outcomes.

References:

- [1] A. Wilansky, *Modern Methods Of Topological Vector Spaces*, (1978), (Book)
- [2] A.A. Hakawati, S.Al-Dwaik , *Best Approximation In Cone Normed Spaces* , An-Najah Journal Of Scientific Research . Vol.30 issue 1, 101 -110, 2016.
<https://doi.org/10.35552/anujr.a.30.1.1175>
- [3] A.A.Hakawati , H.D.Sarries , Cone metric spaces are usual metric spaces by a renorming in the Banach space. *Journal of Nonlinear Analysis and Application*, 2016(2), 106-110 .
<https://doi.org/10.5899/2016/jnaa-00302>
- [4] Alamri, Badriah, and Jamshaid Ahmad. "Fixed point results in b-metric spaces with applications to integral equations." *AIMS Mathematics* 8.4 (2023): 9443-9460.
<https://doi.org/10.3934/math.2023476>
- [5] Alzumi, Hala, and Jamshaid Ahmad. "Fixed-Point Results for Multi-Valued Mappings in Topological Vector Space-Valued Cone Metric Spaces with Applications." *Axioms* 13.12 (2024): 841.
<https://doi.org/10.3390/axioms13120841>
- [6] Amiri, Pari, and Hojjat Afshari. "Common fixed point results for multi-valued mappings in complex-valued double controlled metric spaces and their applications to the existence of solution of fractional integral inclusion systems." *Chaos, Solitons & Fractals* 154 (2022): 111622.
<https://doi.org/10.1016/j.chaos.2021.111622>
- [7] Beg, Ismat, Gunaseelan Mani, and Arul Joseph Gnanaprakasam. "Fixed Point of Orthogonal F-Suzuki Contraction Mapping on O-Complete b-Metric Spaces with applications." *Journal of Function spaces* 2021.1 (2021): 6692112.
<https://doi.org/10.1155/2021/6692112>
- [8] D. Turkoglo, M.Abulooha, T.Abdeljawad, *Some theorems in cone metric spaces*, Journal of Elsevier Science, (18 February 2009), Vol.10.
- [9] D.Turkoglo , M.Abulooha , *Cone metric spaces and fixed point theorems in diametrically contractive mappings* , Acta Mathematica , English series, (2010) 489 – 496 . Vol.26. No.3.
<https://doi.org/10.1007/s10114-010-8019-5>
- [10] D.Turkoglo , M.Abulooha, T.Abdeljawad , *Fixed points of generalized contraction mappings in cone metric spaces* , Math. Commun. 16(2011), 325-334.
<https://hrcak.srce.hr/74876>
- [11] Debnath, Pradip. "Common fixed-point and fixed-circle results for a class of discontinuous F-contractive mappings." *Mathematics* 10.9 (2022): 1605.
<https://doi.org/10.3390/math10091605>
- [12] Francesca Vetro, *Fixed Point Results for Non-Expansive Mappings on Metric Spaces*, Filomat 29:9 (2015), 2011–2020, DOI 10.2298 / FIL1509011V.
<https://doi.org/10.2298/FIL1509011V>
- [13] Fukaya, Tomohiro, and Takumi Matsuka. "Boundary of free products of metric spaces." *arXiv preprint arXiv:2402.06862* (2024).
<https://doi.org/10.48550/arXiv.2402.06862>
- [14] Górka, Przemysław, and Paweł Lefelbajn. "CONE PROPERTY AND MEASURE

- DENSITY CONDITION.*" *Mathematical Inequalities & Applications* 27.2 (2024).
<https://doi.org/10.7153/mia-2024-27-23>
- [15] H. P. Masiha, F. Sabetghadam and A. H. Sanatpour, *Fixed point theorems in cone metric spaces*, The 18th Seminar on Mathematical Analysis and its Applications 26-27 Farvardin, 1388 (15-16 April, 2009) pp. 247-250 Tarbiat Moallem University.51
<https://doi.org/10.1155/2009/125426>
- [16] Haghi, R.H. and Sh. Rezapour. *Fixed points of multifunctions on regular cone metric spaces*. *Expositiones Mathematicae*, 28 (1): 71- 77. (2010).
<https://doi.org/10.1016/j.exmath.2009.04.001>
- [17] Haitham Darweesh., *Cone Metric Space* (Master's thesis, An – Najah University , Palestine) (2016)
- [18] Jeyaraman, M., and S. Sowndrarajan. "*Common fixed point results in neutrosophic metric spaces.*" *Neutrosophic Sets and Systems* 42 (2021): 208-220.
- [19] K. Deimling. *Nonlinear Functional Analysis*. Springer-Verlage, 1985.
<https://doi.org/10.1007/978-3-662-00547-7>
- [20] K. P. R. SASTRY, Ch. S. RAO, A. C. SEKHAR and M. BALAIAH, *On non metrizable cone metric spaces*, *Int. J. of Mathematical Sciences and Applications*, Vol. 1, No. 3, September (2011), 54C60.
- [21] Karapinar, Erdal, et al. "*Fixed-point results for Meir–Keeler type contractions in partial metric spaces: a survey.*" *Mathematics* 10.17 (2022): 3109.
<https://doi.org/10.3390/math10173109>
- [22] Karimizad, Seyedeh Sara, and Ghasem Soleimani Rad. "*Pre-symmetric w-cone distances and characterization of TVS-cone metric completeness.*" *Mathematics* 12.12 (2024): 1833.
<https://doi.org/10.3390/math12121833>
- [23] L. G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, *J. Math. Anal. Appl.* 332 (2) (2007), 1468–1476.
<https://doi.org/10.1016/j.jmaa.2005.03.087>
- [24] Liu, H., & Xu, S. (2013). *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*. *Fixed point theory and applications*, 2013, 1-10.
<https://doi.org/10.1186/1687-1812-2013-320>
- [25] Light, W. and Cheney, W. *Approximation Theory in Tensor Product Spaces*, *Lecture Notes in Math.* 1169, Springer- Verlag, berlin, (1985).
- [26] Limaye, B.V., *Functional Analysis*, Wiley Eastern limited, 1981.
- [27] M. Asadi, S.M.Vaezpour , H.Soleimani , *Metrizability of cone metric spaces* , (2011) arXiv: 1102.2353.v1[math. FA].
<https://doi.org/10.48550/arXiv.1102.2353>
- [28] M. Asadi, S.M.Vaezpour , H.Soleimani , *Metrizability of cone metric spaces via renorming the Banach spaces* . *Nonlinear Analysis and applications*, (2012), Vol.2012, year2012. Article ID jnaa – 00160, 5 pages.
<https://doi.org/10.5899/2012/jnaa-00160>
- [29] M. E. Gordji, M. Ramezani, H. Khodaei and H. Baghani, *Cone Normed Spaces*, arXiv:0912.0960v1 [math. FA] 4 Dec 2009.
<https://doi.org/10.48550/arXiv.0912.0960>

- [30] M. Khani, M. Pourmahdian, *On the metrizable of cone metric spaces*, Topology Appl. 158 (2) (2011), 190–193.
<https://doi.org/10.1016/j.topol.2010.10.016>
- [31] M. Asadi, H. Soleimani, *Examples in Cone Metric Spaces : A Survey*, arXiv 102.4675v1 [math.FA] 23 Feb (2012) .
<https://doi.org/10.5829/idosi.mejsr.2012.11.12.1462>
- [32] M. Ruzhansky, J. Wirth, *Progress in Analysis and Its Applications*, World Scientific, 2010, 9789814313162 . <https://doi.org/10.1007/s10688-013-0010-3>
- [33] Mani, Gunaseelan, et al. "Fixed Point Results in C^* -Algebra-Valued Partial b-Metric Spaces with Related Application." *Mathematics* 11.5 (2023): 1158.
<https://doi.org/10.3390/math11051158>
- [34] Nie, Xin. "On circle patterns and spherical cone al metrics." *Proceedings of the American Mathematical Society* 152.02 (2024): 843-853.
<https://doi.org/10.1090/proc/16557>
- [35] Pant, Rajendra, and Rahul Shukla. "New fixed point results for Proinov–Suzuki type contractions in metric spaces." *Rendiconti del Circolo Matematico di Palermo Series* 271.2 (2022): 633-645. <https://doi.org/10.1007/s12215-021-00649-z>
- [36] Raji, Muhammed, and Musa Adeku Ibrahim. "Fixed point theorems for modified F-weak contractions via α -admissible mapping with application to periodic points." *Annals of Mathematics and Computer Science* 20 (2024): 82-97.
<https://doi.org/10.56947/amcs.v20.232>
- [37] Rasham, Tahair, et al. "A study of some new multivalued fixed point results in a modular like metric space with graph." *The Journal of Analysis* (2022): 1-12.
<https://doi.org/10.1007/s41478-021-00372-z>
- [38] Rashid, M. H. M. "A common fixed point theorem for two self-mappings defined on strictly convex probabilistic cone metric space." *arXiv preprint arXiv:2409.15482* (2024).
<https://doi.org/10.48550/arXiv.2409.15482>
- [39] Rezapour, Sh. and R. Hambarani, (2008). *Some Notes on the Paper Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings*. J. Math. Anal. Appl., 345: 719-724.
- [40] Romaguera, Salvador. "Basic contractions of Suzuki-type on quasi-metric spaces and fixed point results." *Mathematics* 10.21 (2022): 3931. <https://doi.org/10.3390/math10213931>
- [41] Rudin, W. *Real and Complex Analysis*, McGraw-Hill, New York, (1966)
- [42] S. Al-Dwaik, The S – property and best approximation (Master's thesis, An – Najah University, Palestine), (2000) .
- [43] T. Abdeljawad, *Completion of cone metric spaces*, Hacettepe journal of mathematics and statistics, (2010), Vol.39(1), 67 – 74. 50
- [44] T. Abdeljawad, D. Turkoglu, M. Abuloha, *Some theorems and examples of cone Banach spaces*, J. Computational Anal. and App. Vol. 12. No.4. (2010), 739 – 753.
<https://www.researchgate.net/publication/268606423>
- [45] T. Abdeljawad, E. Karapnar, *Quasicone metric spaces and generalizations of Caristi Kirk's Theorem*, Hindawi Publ. Corp. Fixed Point Theory and Applications, (2009), Article ID 574387. <https://doi.org/10.1155/2009/574387>
- [46] Wei-Shih Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal. TMA 72 (2010), 2259–2261.

- [47] Y. Feng, W. Mao, *The equivalence of cone metric spaces and metric spaces*, Fixed Point Theory 11 (2) (2010), 259–264.
<https://www.researchgate.net/publication/268606292>
- [48] Z. Kadelburg and S. Radanovich, *a note on various types of cones and fixed point results in cone metric spaces*, Asian Jour. of Math. Appl., 2013 (2013), Article ID ama0104, pages.
- [49] Z. Kadelburg, S. Radenović, V. Rakoćević, *A note on equivalence of some metric and Cone metric fixed point results*, Appl. Math. Lett. 24 (2011), 370–374
<https://doi.org/10.1016/j.aml.2010.10.030>
- [50] Zhonga, Lingsheng, and Zhongbao Tanga. "Some properties of I-convergence in cone metric spaces." *Filomat* 38.16 (2024): 5819-5826.
<https://doi.org/10.2298/FIL2416819Z>