The Scholar Journal for Sciences & Technology sjst

ISSN (Print) 2794-7629
ISSN (Online) 2794- 4549

Received 02/11/2024
Accepted 01/12/2024

FULL PAPER

Toeplitz Operators on Finite Dimension Spaces with Truncated Values

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Abstract

This work discusses matrix representations of Toeplitz operators' compressing to finite dimensional model spaces $H^2 \ominus BH^2$, where the Blaschke product B is finite. We use the matrix representation to determine if a linear transformation on $H^2 \ominus BH^2$compresses the Toeplitz operator, and if so, what are the sufficient and necessary conditions for this to happen? This result validates a related observation made by Sarason (D. Sarason 1994, D. Sarason2007).

Key words :
Clark operators, representations, model spaces, matrix and Toeplitz operators

1. Introduction

If $L^2 = L^2(\partial\mathbb{D}, d\theta/2\pi)$ is projected orthogonally onto H^2 by P, where H^2 is an open unit disk, $\mathbb{D}:=\{|z|<1\}$, is standard Hardy space, refer to (P. L. Duren 1970). for the fundamental definitions), T_φ is the Toeplitz operator defined on H^2, for $\varphi \in L^\infty$ using the following formula: $T_\varphi f = P(\varphi f)$. A recent investigation into truncated Toeplitz operators was started by Sarason (N. A. Sedlock. (2011). On the model spaces $K_\vartheta := H^2 \cap (\vartheta H^2)^\perp$, these operators $A_\varphi$are defined, where $A_\varphi f := P_\vartheta(\varphi f)$, where ϑ is an inner function. In this case, L^2 is orthogonally projected onto K_ϑ by P_ϑ. Written otherwise, A_φ represents the compression of T_φ to K_ϑ. The set in (N. A. Sedlock. 2011).:

$$\mathcal{T}_\vartheta := \{A_\varphi : \varphi \in L^2 \quad , A_\varphi \text{ is limited}\}$$

is described as follows: When functions $g_1, g_2 \in K_\vartheta$exist, then a bounded operator A on K_ϑ belongs to$\mathcal{T}_\vartheta$

$$A = A_z^* A A_z + g_1 \otimes k + k \otimes g_2, \tag{1}$$

Where $k(z) := \frac{\vartheta(z)-\vartheta(0)}{z}$,and the rank-one operator is shown by

$$h_1 \otimes h_2.\, h_1 \otimes h_2(f) = \langle f, h_2\rangle h_1.$$

The condition in (1) is difficult to develop because it relies on the presence of the functions $g_1, g_2 \in K_\vartheta$, which establish which bounded operators on $K_\vartheta \in \mathcal{T}_\vartheta$. In the case where K_ϑ is finite dimensional, we will define a more explicit condition. A limitless n-dimensional model space is defined as K_B, where B is a finite Blaschke product with zeros $\{a_1,\dots,a_n\}$.It's generally knowledge that all functions with the type $K_B$are are constructed of

$$f(z)=\frac{p(z)}{\prod_{j=1}^{n}(1-\overline{a_j}z)} \qquad (2)$$

where p can be any polynomial with a maximum degree of $n-1$. Additionally,

$$k_\lambda(z):=\frac{1-\overline{B(\lambda)}B(z)}{1-\overline{\lambda}z} \qquad (3)$$

is the K_B resembling kernel such that $k_\lambda\in K_B$,for each $\lambda\in\mathbb{D}$

$$f(\lambda)=\langle f,k_\lambda\rangle \quad \forall f\in K_B.$$

The formula above uses L^2 inner product. $\langle f,g\rangle=\int_{\mathbb{T}}f(\zeta)\overline{g(\zeta)}\frac{|d\zeta|}{2\pi}$, where $T:=\partial\mathbb{D}$. Using (2) and interpolating, we can easily demonstrate that the set $\{k_{\lambda_1},\dots,k_{\lambda_n}\}$ serves as a basis for K_B for multiple points $\lambda_1,\dots,\lambda_n\in\mathbb{D}$. If the zeros $a_1,\dots,a_n$ of B are distinct, $\{k_{a_1},\dots,k_{a_n}\}$ forms a non-orthonormal basis for K_B. $k_{a_j}(z)=\frac{1}{1-\overline{a_j}z}$.

Dimensional n^2 defines the complex vector field of all linear transformations on K_B, given in fundamental linear algebra. The dimension of $\mathcal{T}_B$ is $2n-1$,by Sarason (P. L. Duren 1970). Naturally, this raises the question of which linear transformations on $K_B\in\mathcal{T}_B$. This is the first theorem that we have.

Remark (1.1)

(1) The principal horizontal values and first row form a matrix representing a truncated Toeplitz operator according to Theorem (1.8). Observe that since $\mathcal{T}_B$ has size $2n-1$, similar matrices should also have dimension $2n-1$.

(2) The first row is not particularly noteworthy. An analogous outcome, for instance, can be achieved if the first column and entries along the major diagonal define the representation matrix.

(3) The demonstration of this theorem additionally includes an algorithm for generating the symbol φ from matrix elements.

(4) When $n=2$, the distribution matrix $\begin{pmatrix} r_{1,1} & r_{1,2}\\ r_{2,1} & r_{2,2}\end{pmatrix}$ represents the truncated Toeplitz operator in terms of the basis $\{k_{a_1},k_{a_2}\}$ if and only if $\overline{B'(a_1)}r_{1,2}=r_{2,1}\overline{B'(a_2)}$. $\{k_{a_1},k_{a_2}\}$ is a valid basis for K_B, although it is not orthonormal. The Clark basis $\{v_{\zeta_1},\dots,v_{\zeta_n}\}$, is an essential orthonormal basis for K_B. It consists of normalizing vectors that correspond to the eigenvalues $\zeta_j\in\mathbb{T}$ for the Clark unitary operator U_α where $\alpha\in\mathbb{T}$. This takes the following shape: Because B is an open area of $\mathbb{D}^-$with a finite Blaschke product, the kernel function k_ζ gives the analytical function on $\mathbb{D}$ for any $\zeta\in\mathbb{T}$. Typically, it is shown that $k_\zeta\in K_B$B and:

$$f(\zeta)=\langle f,k_\zeta\rangle \quad \forall f\in K_B. \qquad (4)$$

Using the assumption that B' never disappears on $\mathbb{T}$, a routine exercise will demonstrate that for any $\alpha\in\mathbb{T}$, there are precisely n different locations $\zeta_1,\dots,\zeta_n\in\mathbb{T}$ for which:

$$B(\zeta_j)=\frac{\alpha+B(0)}{1+\overline{B(0)}\alpha},\ \ j=1,\dots,n.$$

One more standard workout will demonstrate that $\|k_{\zeta_s}\|^2 = |B'(\zeta_s)|$, hence, the normalized kernel functions are formed.

$$v_{\zeta_s} := \frac{k_{\zeta_s}}{\sqrt{|B'(\zeta_s)|}}. \tag{5}$$

It shows out that the eigenvalues of the Clark unitary operator are the points $\zeta_1, \dots, \zeta_n$.

$$U_\alpha := A_z + \frac{B(0)+\alpha}{1-|B(0)|^2}\ \left(k_0 \otimes \tilde{k}_0\right) \tag{6}$$

$v_{\zeta_1}, \dots, v_{\zeta_n}$, the appropriate eigenvectors.

$$\tilde{k}_\lambda(z) := \frac{B(z)-B(\lambda)}{z-\lambda}. \tag{7}$$

It is observable (P. L. Duren1970). that for any $\tilde{k}_\lambda \in K_B$ for all $\lambda \in \mathbb{D}$. Therefore, K_B. has an orthonormal basis of $\{v_{\zeta_1}, \dots, v_{\zeta_n}\}$. In-depth research and generalization have been done on the operators U_α, which Clark (N. A. Sedlock. 2011). Initially examined (D. Sarason, John 1994 and Bu, Q., Chen, Y. & Zhu, S. 2021). The matrix shows U_α with regard to this basis is diag $(\zeta_1, \dots, \zeta_n)$, as determined by the spectral theorem. Our subsequent theorem substitutes the kernel functions' basis $\{k_{a_1}, \dots, k_{a_n}\}$ with the Clark basis $\{v_{\zeta_1}, \dots, v_{\zeta_n}\}$.

Theorem (1.2): Take B be the finite Blaschke product of degree n having $\alpha \in \mathbb{T}$. A can be used to denote any transform that is linear in the space with n -dimensions K_B.

A is only a part of $A \in \mathcal{T}_B$ if and only if $M_A = (r_{i,j})$,the multidimensional representation of A in relation to the Clark basis $\{v_{\zeta_1}, \dots, v_{\zeta_n}\}$ corresponding to α.

$$r_{i,j} = \frac{\sqrt{|B'(\zeta_1)|}}{\zeta_j - \zeta_i}\left(\frac{\zeta_j}{\zeta_i}\frac{1}{\sqrt{|B'(\zeta_j)|}}(\zeta_1-\zeta_i)r_{1,i} + \frac{1}{\sqrt{|B'(\zeta_i)|}}(\zeta_j-\zeta_1)r_{1,j}\right) \tag{8}$$

all $1 \le i,j \le n$, and $i \ne j$.

Remark (1.3): As in the previous theorem, the elements along the main diagonal and the first row determine the matrix representation of a truncated Toeplitz operator.

(1) An algorithm for obtaining the symbol φ from the matrix entries will also result from the proving of this theorem.

(2) If n is 2, the matrix $\begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$ is the truncated Toeplitz operator's matrix representation with regard to basis $\{v_{\zeta_1}, v_{\zeta_2}\}$ if and only if

$$\zeta_1 r_{1,2} = \zeta_2 r_{2,1}.$$

If we alter the basis $\{v_{\zeta_1}, \dots, v_{\zeta_n}\}$ slightly, we get even more. Indeed, let

$$\beta_\alpha := \frac{\alpha + B(0)}{1+\overline{B(0)}\alpha},\quad w_s := e^{-\frac{i}{2}(\arg(s)-\arg(\beta_\alpha))},\quad e_{\zeta_s} := \frac{1}{\sqrt{|B'(\zeta_s)|}} w_s k_{\zeta_s}. \tag{9}$$

$\{e_{\zeta_1},\dots,e_{\zeta_n}\}$ is an orthonormal basis that, has the additional characteristic that, in addition to diagonalizing the Clark operator U_α, the matrix representation of any truncated Toeplitz operator with respect to this basis is complex symmetric, as shown by Garcia and Putinar in (Bu, Q., Chen, Y. & Zhu, S. 2021). If a matrix $M = M^t$, where t is the transpose, then M is complex symmetric. The Clark basis $\{v_{\zeta_1},\dots,v_{\zeta_n}\}$ is replaced with the new basis $\{\{e_{\zeta_1},\dots,e_{\zeta_n}\}$ by the theorem that follows.

Theorem (1.4): Assume that B is a finite Blaschke product with $\alpha \in \mathbb{T}$,and degree n.

A can be used to represent any linear transformation on the n-dimensional space K_B.

$A \in \mathcal{T}_B$ if and only if M_A is complex symmetric and $M_A = (r_{i,j})$ represents A as a matrix with respect to the basis $\{e_{\zeta_1},\dots,e_{\zeta_n}\}$that corresponds to α .

$$r_{i,j} = \frac{\sqrt{|B'(\zeta_1)|}}{\overline{w_1}}\frac{1}{\zeta_j-\zeta_i}\left(\frac{\overline{w_j}}{\sqrt{|B'(\zeta_j)|}}(\zeta_1-\zeta_i)+\frac{\overline{w_i}}{\sqrt{|B'(\zeta_i)|}}(\zeta_j-\zeta_1)r_{1,j}\right) \qquad (10)$$

for all $1 \le i,j \le n$, $i \ne j$.

Proof: we prove theorem (1.2) and theorem (1.4) together:

Assume $\alpha_1 \ne \alpha_2$ and fix $\alpha_1,\alpha_2 \in \mathbb{T}$. Let the points of $\mathbb{T}$ that $\{\zeta_j,\eta_j : j = 1,\dots,n\}$are those that $B(\zeta_j) = \beta_{\alpha_1}$, $B(\eta_j) = \beta_{\alpha_2}$, $j = 1,\dots,n$. Remark (1.11) informs us that any member of $\mathcal{T}_B$ adopts the form $\sum_{j=1}^{n} c_j k_{\zeta_j} \otimes k_{\zeta_j} + \sum_{j=1}^{n} d_j k_{\eta_j} \otimes k_{\eta_j}$, regarding a few complex constants, c_j, d_j. Assume that $e_{\zeta_s} = w_s v_{\zeta_s}$, where:

$$w_s = e^{-\frac{i}{2}(\arg(\zeta_s)-\arg(\beta_{\alpha_1}))}. \qquad (11)$$

Theorem (1.4) can be established by using

$\langle\left(\sum_{j=1}^{n} c_j k_{\zeta_j} \otimes k_{\zeta_j} + \sum_{j=1}^{n} d_j k_{\eta_j} \otimes k_{\eta_j}\right) e_{\zeta_p}, e_{\zeta_s}\rangle$,the operator's matrix representation with regard to the basis $\{e_{\zeta_1},\dots,e_{\zeta_n}\}$, Since $\{e_{\zeta_1},\dots,e_{\zeta_n}\}$ is an orthonormal basis for K_B, the \`Fourier' expansion is available for every $f \in K_B$.

$$f(z) = \sum_{s=1}^{n} \langle f, e_{\zeta_s}\rangle e_{\zeta_s}(z) = \sum_{s=1}^{n} \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} f(\zeta_s) e_{\zeta_s}(z)$$

and so

$$\langle f, g\rangle = \sum_{s=1}^{n} \frac{f(\zeta_s)\overline{g(\zeta_s)}}{\sqrt{|B'(\zeta_s)|}}, \quad f, g \in K_B. \qquad (12)$$

First notice that

$$e_{\zeta_s}(\zeta_q) = \frac{w_s}{\sqrt{|B'(\zeta_s)|}} k_{\zeta_s}(\zeta_q) = \begin{cases} w_s\sqrt{|B'(\zeta_s)|}, & \text{if } s = q; \\ 0, & \text{if } s \ne q \end{cases}.$$

Using the inner product formula given before in (12), we get

$$\langle\left(k_{\zeta_j\otimes}k_{\zeta_j}\right)e_{\zeta_p},e_{\zeta_s}\rangle=\sum_{q=1}^{n}\frac{\left(\left(k_{\zeta_j}\otimes k_{\zeta_j}\right)e_{\zeta_p}\right)(\zeta_q)\overline{e_{\zeta_s}(\zeta_q)}}{\sqrt{|B'(\zeta_q)|}}=\left(\left(k_{\zeta_j}\otimes k_{\zeta_j}\right)e_{\zeta_p}\right)(\zeta_s)\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}$$

$$=\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}\langle e_{\zeta_p},k_{\zeta_j}\rangle k_{\zeta_j}(\zeta_s)$$

$$=\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}\frac{w_p}{\sqrt{|B'(\zeta_p)|}}k_{\zeta_p}(\zeta_j)k_j(\zeta_j)=\begin{cases}|B'(\zeta_s)|, & \text{if } s=p=j;\\ 0, & \text{otherwise.}\end{cases}$$

In a similar way,

$$\langle\left(k_{\eta_j}\otimes k_{\eta_j}\right)e_{\zeta_p},e_{\zeta_s}\rangle=\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}\frac{w_p}{\sqrt{|B'(\zeta_p)|}}k_{\zeta_p}(\eta_j)k_{\eta_j}(\zeta_s)$$

$$=\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}\frac{w_p}{\sqrt{|B'(\zeta_p)|}}\frac{1-\overline{B(\zeta_p)}B(\eta_j)}{1-\overline{\zeta_p}\eta_j}\frac{1-\overline{B(\eta_j)}B(\zeta_s)}{1-\overline{\eta_j}\zeta_s}$$

$$=\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}\frac{w_p}{\sqrt{|B'(\zeta_p)|}}\frac{1-\overline{\beta_{\alpha_1}}\beta_{\alpha_2}}{1-\overline{\zeta_p}\eta_j}\frac{1-\overline{\beta_{\alpha_2}}\beta_{\alpha_1}}{1-\overline{\eta_j}\zeta_s}$$

$$=\left|1-\overline{\beta_{\alpha_2}}\beta_{\alpha_1}\right|^2\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}\frac{w_p}{\sqrt{|B'(\zeta_p)|}}\frac{1}{1-\overline{\zeta_p}\eta_j}\frac{1}{1-\overline{\eta_j}\zeta_s}$$

$$=\left|1-\overline{\beta_{\alpha_2}}\beta_{\alpha_1}\right|^2\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}\frac{w_p}{\sqrt{|B'(\zeta_p)|}}\frac{1}{1-\overline{\zeta_p}\eta_j}\frac{1}{1-\overline{\eta_j}\zeta_s}\frac{(-\eta_j)\zeta_p}{(-\eta_j)\zeta_p}$$

$$=-\eta_j\left|1-\overline{\beta_{\alpha_2}}\beta_{\alpha_1}\right|^2\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}\frac{w_p\zeta_p}{\sqrt{|B'(\zeta_p)|}}\frac{1}{\eta_j-\zeta_s}\frac{1}{\eta_j-\zeta_p}$$

Using (11) is notion of w_p, the true nature, $\zeta_p=\beta_{\alpha_1}\overline{w_p}^2$. Apply this characteristic to change the previous expression's final line in order to

$$-\beta_{\alpha_1}\eta_j\left|1-\overline{\beta_{\alpha_2}}\beta_{\alpha_1}\right|^2\frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}}\frac{\overline{w_p}}{\sqrt{|B'(\zeta_p)|}}\frac{1}{\eta_j-\zeta_s}\frac{1}{\eta_j-\zeta_p}.$$

Putting this all together, we get

$$\langle\left(\sum_{j=1}^{n} c_j k_{\zeta_j} \otimes k_{\zeta_j} + \sum_{j=1}^{n} d_j k_{\eta_j} \otimes k_{\eta_j}\right) e_{\zeta_p}, e_{\zeta_s}\rangle$$

$$= c_p\left|B'(\zeta_p)\right|\delta_{s,p} - \beta_{\alpha_1}\left|1-\overline{\beta_{\alpha_2}}\beta_{\alpha_1}\right|^2 \sum_{j=1}^{n} \eta_j d_j \frac{\overline{w_s}}{\sqrt{|B'(\zeta_s)|}} \frac{\overline{w_p}}{\sqrt{\left|B'(\zeta_p)\right|}} \frac{1}{\eta_j - \zeta_s} \frac{1}{\eta_j - \zeta_p}.$$

Partial fraction decomposition is used.

$$\frac{1}{\eta_j - \zeta_s}\frac{1}{\eta_j - \zeta_p} = \frac{1}{\zeta_s - \zeta_p}\left(\frac{1}{\eta_j - \zeta_s} - \frac{1}{\eta_j - \zeta_p}\right),$$

The identities in can be confirmed in (10). The requirements in (10), are then satisfied by any truncated Toeplitz operator's matrix representation with respect to the basis $\{e_{\zeta_1}, \dots, e_{\zeta_n}\}$. The inverse proof is almost identical to the converse proof in Theorem (1.8). By employing comparable computations to the Theorem (1.4) proof, it is demonstrated that

$$\langle\left(\sum_{j=1}^{n} c_j k_{\zeta_j} \otimes k_{\zeta_j} + \sum_{j=1}^{n} d_j k_{\eta_j} \otimes k_{\eta_j}\right) v_{\zeta_p}, v_{\zeta_s}\rangle =$$

$$c_p\left|B'(\zeta_p)\right|\delta_{s,p} - \beta_{\alpha_1}\left|1-\overline{\beta_{\alpha_2}}\beta_{\alpha_1}\right|^2 \sum_{j=1}^{n} \eta_j d_j \frac{1}{\sqrt{|B'(\zeta_s)|}} \frac{\zeta_p}{\sqrt{\left|B'(\zeta_p)\right|}} \frac{1}{\eta_j - \zeta_s} \frac{1}{\eta_j - \zeta_p}.S$$

Proceed with the remaining steps in Theorem (1.4) proof to establish Theorem (1.2).

Remark (1.5): Any complex symmetric 2×2 matrix represents a truncated Toeplitz operator with regard to the basis $\{e_{\zeta_1}, e_{\zeta_2}\}$, according to the theorem for $n = 2$. Sarason had noted this before (R.Garcia and M.Putinar, August 2007 & Balayan, L., Garcia, S.R., 2010).
Sarason started talking about how the truncated Toeplitz operators are generated by the Clark unitary operators in some way in Remark (1.10) below, (N. A. Sedlock. 2011). In finite dimensions, the outcome is as follows.

Theorem (1.6): Assume that $\alpha_1, \alpha_2 \in \mathbb{T}$, where $\alpha_1 \neq \alpha_2$, and that a Blaschke product of degree n is represented by B. Next, for any $\varphi \in L^2$, there exist polynomials $p; q$ with a maximum degree of n, such that

$$A_\varphi = p(U_{\alpha_1}) + q(U_{\alpha_2}). \qquad (13)$$

Proof: One can obtain the ensuing lemma from (P. L. Duren (1970). Here we provide evidence.

Remark (1.7):

(1) Sarason (P. L. Duren 1970). establishes that for any polynomial p and each $\alpha \in \mathbb{T}$, $p(U_\alpha) \in \mathcal{T}_B$. In reality, the spectral theorem for unitary operators and Theorem (1.6) may be extracted from the proof in (N. A. Sedlock. 2011).
(2) Remark (1.9) will demonstrate that the polynomials p and q in (13) can be computed, in a sense, from φ.

Theorem (1.8): Let B be a finite Blaschke product of degree n with unique zeros $a_1, \dots, a_n$, and let A be any linear transformation on the n-dimensional space K_B. A is represented by the matrix $M_A = (r_{i,j})$with respect to the basis $\{k_{a_1}, \dots, k_{a_n}\}$if and only if $A \in \mathcal{T}_B$.

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$$r_{i,j}=\overline{\left(\frac{B'(a_1)}{B'(a_i)}\right)}\left(\frac{r_{1,i}\overline{(a_1-a_i)}+r_{1,j}\overline{(a_j-a_1)}}{\overline{a_j-a_i}}\right),\ \ 1\le i,j\le n,\ \ i\ne j. \qquad (14)$$

Proof: Given a φ that is in L^2, break φ down as

$$\varphi=\psi_1+\overline{\psi_2}+\eta_1+\overline{\eta_2},\ \ \psi_1,\psi_2\in K_B,\ \ \eta_1,\eta_2\in BH^2.$$

Now write A_φ as $A_\varphi=A_{\psi_1+\overline{\psi_2}}+A_{\eta_1+\overline{\eta_2}}$, and see that (P. L. Duren 1970). 's second term, which is zero, is on the right. Thus,

$$\left\{A_{\psi_1+\overline{\psi_2}}:\psi_1,\psi_2\in K_B\right\}=\mathcal{T}_B. \qquad (15)$$

The zeros $a_1,\dots,a_n$ of B are assumed to be different, and as a result, the functions

$\tilde{k}_{a_j}(z)=\frac{B(z)}{z-a_j},\ \ j=1,\dots,n$, as a foundation for K_B serve as a foundation for K_B , and $\overline{\tilde{k}_{a_j}(z)}=\overline{\left(\frac{B(z)}{z-a_j}\right)},\ \ j=1,\dots,n$, form a basis for $\overline{K_B}$. Based on the preceding discourse and equation (15), $\mathcal{T}_B$ is comprised of A_φ.

$$\varphi(\zeta)=\sum_{j=1}^{n}c_j\overline{\left(\frac{B(\zeta)}{\zeta-a_j}\right)}+\sum_{j=1}^{n}d_j\frac{B(\zeta)}{\zeta-a_j} \qquad (16)$$

and the arbitrary complex numbers c_j,d_j. Add this to the identity.

$$k_\lambda\otimes\tilde{k}_\lambda=A_{\frac{B}{\overline{z-\lambda}}}. \qquad (17)$$

And its adjoint to determine that the operators in $\mathcal{T}_B$ are of the following type in (P. L. Duren 1970).

$$\sum_{j=1}^{n}c_jk_{a_j}\otimes\tilde{k}_{a_j}+\sum_{j=1}^{n}d_j\tilde{k}_{a_j}\otimes k_{a_j}, \qquad (18)$$

where the complex integers c_j and $d_j$are. The matrix representation of the previously specified operator about the basis $\{k_{a_1}\dots,k_{a_n}\}$.will be determined shortly. We must first obtain a few formulas. With the definitions of k_{a_j} (3) and $\tilde{k}_{a_j}$) (7) and the replicating property of k_{a_j}, we obtain:

$$\langle\tilde{k}_{a_j},k_{a_j}\rangle=\begin{cases}0, & \text{if } i\ne j;\\ B'(a_j), & \text{if } i=j.\end{cases}\ \ \text{and}\ \langle\tilde{k}_{a_j},\tilde{k}_{a_j}\rangle=\frac{1}{1-\overline{a_j}a_i}. \qquad (19)$$

As $\{k_{a_1},\dots,k_{a_n}\}$ gives a basis for K_B, we understand that ,$\tilde{k}_{a_j}=\sum_{s=1}^{n}h_s(a_j)k_{a_s}$,

$h_s(a_j)$for a number of complex constants. (19) can be used to calculate $h_s(a_j)$and obtain:

$$\tilde{k}_{a_j}=\sum_{s=1}^{n}\frac{1}{\overline{B'(a_s)1}}\frac{1}{1-\overline{a}_sa_j}k_{a_s}. \qquad (20)$$

The proof of Theorem (1.8) is now ready. Assume A_φ has the structure shown in (18) and $\left(b_{s,p}\right)_{1\le s,p\le n} = M_{A_\varphi}$, express A_φ as a matrix with regard to the basis $\{k_{a_1},\dots,k_{a_n}\}$. We need for proof that

$$b_{s,p} = \left(\frac{\overline{B'(a_1)}}{\overline{B'(a_s)}}\right)\left(\frac{b_{1,s}\overline{(a_1-a_s)} + b_{1,p}\overline{(a_p-a_1)}}{\overline{a_p}-\overline{a_s}}\right),\quad 1\le s,p\le n, s\ne p \qquad (21)$$

A computation with (18), (19), and (20) will show that

$$A_\varphi k_{a_p} = c_p\overline{B'(a_p)}k_{a_p} + \sum_{s=1}^{n}\left(\frac{1}{\overline{B'(a_s)}}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a_s}a_j)(1-\overline{a_p}a_j)}\right)k_{a_s}.$$

Thus

$$b_{s,p} = c_p\overline{B'(a_p)}\delta_{s,p} + \frac{1}{\overline{B'(a_s)}}\sum_{j=1}^{n}\frac{d_j}{(1-\overline{a_s}a_j)(1-\overline{a_p}a_j)}.$$

From the formula, the unique characteristics in (21) follow.

$$\frac{1}{(1-\overline{a_s}a_j)(1-\overline{a_p}a_j)} = \frac{-\overline{a_s}}{\overline{a_p}-\overline{a_s}}\frac{1}{1-\overline{a_s}a_j} + \frac{\overline{a_p}}{\overline{a_p}-\overline{a_s}}\frac{1}{1-\overline{a_p}a_j}.$$

The proof has now been completed in one direction. Let V be the set of all matrices that fulfill (14) in the opposite direction. These identities show that each

$M = \left(r_{i,j}\right) \in V$ is uniquely determined by the entries along the diagonal and the first row. Moreover, M is these entries' linear function. V is therefore a vector space with dimensions of $2n-1$. As we've already established in (21),

$V_1 := \left\{M_{A_\varphi} : A_\varphi \in \mathcal{T}_B\right\} \subset V$, and V_1 has dimension $2n-1$ according to Sarason's theorem. $V_1 = V$ as a result, this completes the proof.

Remark (1.9):

(1) Take note that one explicit foundation for V is $\{D_1,\dots,D_n,R_2,\dots,R_n\}$.
Here, R_k is the matrix satisfying equation (14) with $r_{1,k}=1$, $r_{j,j}=0$ if
$j \ne k$, and, $r_{1,j}=0$ for all j. $D_k = \mathrm{diag}(0,\dots,1,0,\dots,0)$ is the matrix. To illustrate, if $n=3$, then

$$D_1 = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad D_2 = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad D_3 = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$R_2 = \begin{pmatrix}0 & 1 & 0\\ \dfrac{B'(a_1)}{B'(a_2)} & 0 & \dfrac{(a_1-a_2)B'(a_1)}{(a_3-a_2)B'(a_2)}\\ 0 & \dfrac{(a_2-a_1)B'(a_1)}{(a_2-a_3)B'(a_3)} & 0\end{pmatrix}$$

$$R_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \dfrac{(a_3 - a_1)B'(a_1)}{(a_3 - a_2)B'(a_2)} \\ \dfrac{B'(a_1)}{B'(a_3)} & \dfrac{(a_1 - a_3)B'(a_1)}{(a_2 - a_3)B'(a_3)} & 0 \end{pmatrix}$$

in the example above indicates the complex conjugation of each and every matrix entry—not the conjugate transpose.

(2) If $P_j := \frac{1}{B'(a_j)} k_{a_j} \otimes \tilde{k}_{a_j}$, take note of the evidence above that

$$P_j^2 = P_j, \ \sum_{j=1}^{n} P_j = I, \ \ P_j P_l = \delta_{j,l} P_j, \ \ \mathcal{T}_B = \mathrm{span}\{P_j, P_j^* : j = 1, \dots, n\}.$$

Comparable identities apply to $P_j = \frac{1}{B'(a_j)} \tilde{k}_{a_j} \otimes k_{a_j}$. These identities show how the set of $2n$ operators $\{P_j, P_j^* : j = 1, \dots, n\}$. has a linear dependence. For instance, a little study are going to show that the basis for $\mathcal{T}_B$, which includes rank one idempotent, is the set $\{P_j, P_l^* : j = 2, \dots, n; l = 1, \dots, n\}$.

(3) A_φ from (18) can be computed with regard to the basis $\{\tilde{k}_{a_1}, \dots, \tilde{k}_{a_n}\}$.using comparable methods. The $b_{s,p}$ element of this matrix is in this instance

$$b_{s,p} = d_p B'(a_p) \delta_{s,p} + \frac{1}{B'(a_s)} \sum_{j=1}^{n} \frac{c_j}{(1 - a_s \overline{a_j})(1 - a_p \overline{a_j})}$$

and the prerequisite that must be met in order for a matrix $(r_{s,p})$ to represent something from $\mathcal{T}_B$ (relative to the basis $\{\tilde{k}_{a_1}, \dots, \tilde{k}_{a_n}\}$) is

$$r_{s,p} = \frac{B'(a_1)}{B'(a_s)} \left(\frac{r_{1,s}(a_1 - a_s) + r_{1,p}(a_p - a_1)}{a_p - a_s} \right), \ \ 1 \le s, p \le n, s \ne p.$$

Lemma (1.10): Assume that $w_1, \dots, w_{2n-1})$ are unique locations within T. Next, the top-ranked operators $k_{w_1} \otimes k_{w_1}, \dots, k_{w_{2n-1}} \otimes k_{w_{2n-1}}$, possess linear independence

Proof: Suppose $c_1, \dots, c_{2n-1}$ are complicated constants in such a way that

$$\sum_{j=1}^{2n-1} c_j k_{w_j} \otimes k_{w_j} = 0. \tag{22}$$

Given the linear independence of $k_{w_1}, \dots, k_{w_n}$, there exists a $g \in K_B$ such that

$$\langle k_{w_1}, g \rangle = 1, \ \ \langle k_{w_j}, g \rangle = 0, \ \ j = 2, \dots, n.$$

Use the operator on the left side of (22) in this case to see that

$$c_1 k_{w_1} + \sum_{j=n+1}^{2n-1} c_j \langle g, k_{w_j} \rangle k_{w_j} = 0.$$

Nevertheless $c_1 = 0$,since the vectors $k_{w_1}, k_{w_{n+1}}, \dots, k_{w_{2n-1}}$ are linearly independent. Select a suitable g now to demonstrate that $c_2 = 0$, and so forth. This is the theorem (1.6) proof. Since $\alpha_1 \neq \alpha_2$, let $\alpha_1, \alpha_2 \in \mathbb{T}$. Let T's points be $\zeta_1, \dots, \zeta_n$ and $\eta_1, \dots, \eta_n$, so that

$$B(\zeta_j) = \beta_{\alpha_1} := \frac{\alpha_1 + B(0)}{1 + \overline{B(0)}\alpha_1}, \quad B(\eta_j) = \beta_{\alpha_2} := \frac{\alpha_2 + B(0)}{1 + \overline{B(0)}\alpha_2}, \quad j = 1, \dots, n.$$

Take in consideration that $\{v_{\zeta_1}, \dots, v_{\zeta_n}\}$is an orthonormal basis for K_B of eigenvectors of U_{α_1}. Observe that the points $\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n$ are different.

Similarly, $\{v_{\eta_1}, \dots, v_{\eta_n}\}$. is an orthonormal basis for K_B of eigenvectors of U_{α_2}. Kindly:

$$P_{\zeta_j} := v_{\zeta_j} \otimes v_{\zeta_j}, \quad P_{\eta_j} := v_{\eta_j} \otimes v_{\eta_j}, \quad j = 1, \dots, n$$

and see that these operators are orthogonal projections onto the eigenspaces they span, respectively, as k_{ζ_j}. It is shown in (D. N. Clark, 1972). that for each } $\zeta \in \mathbb{T}$,

$$k_\zeta \otimes k_\zeta = A_{k_\zeta + \overline{k_\zeta} - 1}$$

Thus, $\mathcal{T}_B$ also includes these projections P_{ζ_j}, P_{η_j}. Also, for any pair of analytic polynomials p and q, we have the following thanks to the spectral theorem for unitary operators:

$$p(U_{\alpha_1}) = \sum_{j=1}^{n} p(\zeta_j) v_{\zeta_j} \otimes v_{\zeta_j}, \quad q(U_{\alpha_2}) = \sum_{j=1}^{n} q(\eta_j) v_{\eta_j} \otimes v_{\eta_j}.$$

and so $p(U_{\alpha_1}), q(U_{\alpha_2}) \in \mathcal{T}_B$.

Then, to show that

$$\mathcal{T}_B = \bigvee \left\{ (U_{\alpha_1})^i, (U_{\alpha_2})^j, \ 1 \leq i, j \leq n \right\},$$

it is sufficient to show that

$$\mathcal{T}_B = \bigvee \left\{ P_j, P_{\eta_j} \ j = 1, \dots, n \right\}.$$

This is inferred immediately from $\mathcal{T}_B$ is dimension of $2n - 1$and Lemma (1.10).

Remark (1.11):

(1) According to Theorem (1.7), for some polynomials p and q, any A_φ has the form $p(U_{\alpha_1}) + q(U_{\alpha_2})$. Here, we note that if we choose the symbol φ carefully, we can infer p and q from it. To see how to accomplish this, take note of how we have demonstrated that in the theorem (1.6) proof.

$$\bigvee \left\{ k_{\zeta_j} \otimes k_{\zeta_j}, k_{\eta_j} \otimes k_{\eta_j} : j = 1, \dots, n \right\} = \mathcal{T}_B.$$

Actually, $\mathcal{T}_B$is basis will be any $2n - 1$ of them. However, given that

$$k_\zeta \otimes k_\zeta = A_{k_\zeta + \bar{k}_{\zeta - 1}},$$

It is possible to write each operator in $\mathcal{T}_B$ as A_φ where

$$\varphi = \sum_{j=1}^{n} c_j \left(k_{\zeta_j} + \overline{k}_{\zeta_{j-1}}\right) + \sum_{j=1}^{n} d_j \left(k_{\eta_j} + \overline{k}_{\eta_{j-1}}\right).$$

Select polynomials p and q with a maximum degree of n.

$$p(\zeta_j) = \sqrt{|B'(\zeta_j)|}c_j, \;\; q(\eta_j) = \sqrt{|B'(\eta_j)|}d_j, \quad j = 1, \dots, n,$$

Then we have

$$A_\varphi = p(U_{\alpha_1}) + q(U_{\alpha_2}).$$

As the spectral theorem indicates

$$p(U_{\alpha_1}) = \sum_{j=1}^{n} p(\zeta_j) v_{\zeta_j} \otimes v_{\zeta_j}, \;\; q(U_{\alpha_2}) = \sum_{j=1}^{n} q(\eta_j) v_{\eta_j} \otimes v_{\eta_j}.$$

Here is the outcome as of right now.

(3) Sarason (P. L. Duren 1970). began a discussion of how the Clark unitary operators provide $\mathcal{T}_\vartheta$ for a generic inner function ϑ. Using the Clark theory and some recent results of Aleksandrov and Poltoratski, he obtained the following integral formula for a limited Borel function φ and an inner function ϑ:

$$A_\varphi = \int_{\mathbb{T}} \varphi(U_\alpha) \frac{|d\alpha|}{2\pi}, \tag{23}$$

If the weak meaning of the aforementioned integral is understood, that is,

$$\langle A_\varphi f, g\rangle = \int_{\mathbb{T}} \langle \varphi(U_\alpha) f, g\rangle \frac{|d\alpha|}{2\pi}, \;\; f, g \in K_\vartheta.$$

There is also a variant of this formula where $\varphi \in L^2$ (not necessarily bounded), although it requires highly specific interpretation. Additionally, Sarason establishes the closure of $\mathcal{T}_\vartheta$ within the topology of weak operators. Is that the case?

$$\mathcal{T}_\vartheta := \bigvee \{q(U_\alpha) : q \text{ is a trigonometric polynomial}, \alpha \in \mathbb{T}\}\,? \tag{24}$$

$\bigvee$ represents the closed linear span in the weak operator topology mentioned above. When the Blaschke product ϑ is finite, then this is unquestionably true (Theorem (1.6)). It is sufficient to demonstrate that $\{A_\varphi : \varphi \in L^\infty\}$ is thick in $\mathcal{T}_\vartheta$ in order to show (24) using (23). As previously stated, it is uncertain if the set above genuinely equals $\mathcal{T}_\vartheta$.

Results:

1- The theorem for $n = 2$ shows that any complex symmetric 2×2 matrix represents a truncated Toeplitz operator with respect to the basis $\{e_{\zeta_1}, e_{\zeta_2}\}$.

2- Really, the proof presented in (P. L. Duren 1970). can be used to develop the spectral theorem for unitary operators and Theorem (1.6).

3- The method in which $\mathcal{T}_{\vartheta}$ is provided for a generic inner function ϑ by the Clark unitary operators was introduced by Sarason (P. L. Duren 1970). He obtained the following integral formula for a limited Borel function φ and an inner function ϑ using the Clark theory and some new results of Aleksandrov and Poltoratski.

Conclusion:

We study Toeplitz operator compressions to coinvariant subspaces of $H^2 \ominus BH^2$. Many characterizations of these operators are found; those of rank one is described in particular. A portion of the material is explanatory. A necessary and sufficient condition has defined to explain the closed and bounded of Blaschke product.

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