

FULL PAPER

Solving Quasilinear Elliptic Equations with Measures using Hölder Continuity

Abstract

. Let's consider a measure μ_i that fulfills the following condition:

$\mu_i(B(x_i, r)) \leq Mr^{n-1+(\alpha-1)\varepsilon}$ for some $\alpha > \beta$ and $\varepsilon > 0$,

and $u_i \in C^{0,\beta_1}(B(x_0, R))$ This is presumed to represent the resolution of the equation:

$$-div(|\nabla u_i|^{-1+\varepsilon} \nabla u_i) = \mu_i.$$

It has been demonstrated that μ_i is locally continuous of β -Hölder if and only if μ_i holds.

Key words: measure, elliptic equations, super harmonic, Hölder continuity

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1. Introduction

We suppose that $\Omega_i \subset \mathbb{R}^n$ denotes an open set and $0 < \epsilon \leq n - 1$ a constant value. Consider $\mu_i \in \Omega_i$ to be a nonnegative Radon type measure, and $u_i \in W_{\text{loc}}^{1,1+\epsilon}$ be a solution of the quasilinear elliptic equation.

Let Ω_i denotes an open set contained in $\mathbb{R}^n$ and let ϵ be a fixed number such that $0 < \epsilon \leq n - 1$. Let $\mu_i \in \Omega_i$ be a measure nonnegative Radon type, and let $u_i \in W_{\text{loc}}^{1,1+\epsilon}$ be the solutions of the quasilinear elliptic equation.

$$-\text{div } \mathcal{A}(x_i, \nabla u_i) = \mu_i. \quad (1.1)$$

Let $\mathcal{E}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote a function that adheres to the following hypotheses with specific values $0 < \lambda \leq \Lambda < \infty$:

$$\begin{cases} \text{For any } \psi_i \in \mathbb{R}^n, \text{ the function } x_i \mapsto \mathcal{E}(x_i, \psi_i) \text{ is measurable, and} \\ \text{For } x_i \in \mathbb{R}^n, \text{ the function } \psi_i \mapsto \mathcal{E}(x_i, \psi_i) \text{ is continuous.} \end{cases} \quad (1.2)$$

for all $\psi_i \in \mathbb{R}^n$ and a.e. $x_i \in \mathbb{R}^n$

$$\mathcal{E}(x_i, \psi_i) \cdot \psi_i \geq \lambda |\psi_i|^{1+\epsilon} \quad (1.3)$$

$$|\mathcal{E}(x_i, \psi_i)| \leq \Lambda |\psi_i|^\epsilon \quad (1.4)$$

$$(\mathcal{E}(x_i, \psi_i) - \mathcal{E}(x_i, \phi_i)) \cdot (\psi_i - \phi_i) > 0 \quad (1.5)$$

whenever $\psi_i \neq \phi_i$. The $(1 + \epsilon)$ -Laplacian

$$-\nabla_{1+\epsilon} u_i = -\text{div}(|\nabla u_i|^{-1+\epsilon} \nabla u_i)$$

is the most prominent illustration of the operators. We examine specific $\mathcal{E}$ -superharmonic functions, which are representatives of solutions. In this context,

It is adequate to ascertain that if u_i constitutes a solution to (1.1) within Ω_i ,

specifically, u_i belongs to $W_{\text{loc}}^{1,1+\epsilon}(\Omega_i)$ and

$$\int_{\Omega_i} \mathcal{E}(x_i, \nabla u_i) \cdot \nabla \vartheta_i dx_i = \int_{\Omega_i} \vartheta_i d\mu_i$$

for each function ϑ_i within the space $C_0^\infty(\Omega_i)$, there exists a unique representation $u_i: \Omega_i \rightarrow (-\infty, \infty]$ that is lower semicontinuous that satisfies

$$u_i(x_i) = \lim_{r \rightarrow 0} \operatorname{ess\,inf}_{B(x_i, r)} u_i$$

for every x_i belonging to the set Ω_i . This individual is referred to as $\mathcal{E}$ -superharmonic. To access additional information on $\mathcal{E}$ -superharmonic functions, refer to references [4] and [5].

2. Main theorem and Basic results:

Theorem 2.1: Let's assume that u_i is $\mathcal{E}$ -superharmonic in Ω_i , where $\mathcal{E}$ denotes a certain property.

$$-\operatorname{div} \mathcal{E}(x_i, \nabla u_i) = \mu_i.$$

Let M be a positive constant and α be a constant between 0 and 1.

$$\mu_i(B(x_i, r)) \leq M r^{n-1+(\alpha-1)\varepsilon} \quad (2.1)$$

For all $\beta \in (0, \min(\alpha, \kappa))$, then $u_i \in C^{0, \beta}(\Omega_i)$ whenever $\kappa = \kappa(n, \varepsilon, \lambda, \Lambda) > 0$.

This is true whenever $B(x_i, 3r) \subset \Omega_i$.

The value κ in Theorem 1.1 represents the optimal local Hölder continuity exponent for $\mathcal{E}$ -harmonic functions. In other words, it characterizes the highest degree of smoothness that continuous solutions of the equation can possess.

$$-\operatorname{div} \mathcal{E}(x_i, \nabla u_i) = 0.$$

See Lemma 2.3, presented below. For the $(1 + \varepsilon)$ -Laplacian, we can set $k = 1$ and achieve the following: we refer to $\mathcal{E}$ -harmonic and $\mathcal{E}$ -superharmonic functions as $(1 + \varepsilon)$ -harmonic and $(1 + \varepsilon)$ -superharmonic functions, respectively.

Corollary 2.2: Let u_i be $(1 + \varepsilon)$ -superharmonic in Ω_i with

$$-\Delta_{1+\varepsilon} u_i = \mu_i$$

and (2.1) holds with $0 < \alpha \leq 1$. Then $u_i \in C^{0, \beta}(\Omega_i)$ for all $\beta \in (0, \alpha)$.

It is straightforward to understand that (1.6) is true if the solution to (1.1) is α -Hölder continuous on Ω_i ; ([6]) or Remark 2.6 below. Then, u_i is a locally Hölder continuous that has a positive exponent under the condition (2.2) as is now widely recognized ([6], [9], [12]). As a novelette, we can acquire a Hölder continuity exponent that is close to α as we desire. If $\varepsilon = 0$ and we are in the

ordinary Laplacian situation, the outcome is classical: (refer to [8] or [13]). The equation (1.1) could be made to depend on u_i itself by employing conventional freezing arguments. This is evident.

It is worth mentioning that Lieberman, as referenced in [9], demonstrated that a solution to

$$-\Delta_{2+\varepsilon} u_i = \mu_i$$

If (2.2) is true for some $\alpha > 1$, then Ω_i belongs to $C^{1,\beta}(\Omega_i)$ for some $\beta > 1$.

In the initial lemma, we obtain the value of k .

Lemma 2.3: There is a positive number $\kappa = \kappa(n, \varepsilon, \lambda, \Lambda) > 0$ for each $0 < r < R$ and $\mathcal{E}$ -harmonic function h in $B(x_0, R)$, the following inequality is satisfied:

$$\int_{B(x_0, r)} |\nabla h|^{1+\varepsilon} dx_i \leq \tilde{c} \left(\frac{r}{R}\right)^{n-(1+\varepsilon)+(1+\varepsilon)\kappa} \int_{B(x_0, R)} |\nabla h|^{1+\varepsilon} dx_i$$

where $\tilde{c} = \tilde{c}(n, \varepsilon, \lambda, \Lambda) > 0$.

Furthermore, the value of $\kappa(n, \varepsilon, 1, 1) = 1$ is equal to 1. Specifically, for the $(1 + \varepsilon)$ -Laplacian, we can select $\kappa = 1$.

Proof: Let κ be a positive number determined by the variables n, ε, λ , and Λ . There exists a constant $\tilde{c}$, also dependent on n, ε, λ , and Λ , which is greater than zero, and let:

$$\text{osc}(h, B(x_0, r)) \leq \tilde{c} \left(\frac{r}{R}\right)^\kappa \text{osc}(h, B(x_0, R)) \quad (2.2)$$

Whenever the function h is $\mathcal{E}$ -harmonic within the ball $B(x_0, R)$, for all values of r satisfying $0 < r < R$. There exists a number κ ([4]). For the $(1 + \varepsilon)$ -Laplacian, we can use $k = 1$ in equation (2.2) under the assumption that $r < R/4$. By employing the gradient approximation (see [10]) for $(1 + \varepsilon)$ -harmonic functions alongside the Caccioppoli inequality (see [3]), we can deduce that

$$\text{osc}(u_i, B(x_0, r))^{1+\varepsilon} \leq (2r)^{1+\varepsilon} \sup_{B(x_0, R/4)} |\nabla h|^{1+\varepsilon} \leq \frac{\tilde{c} r^{1+\varepsilon}}{R^n} \int_{B(x_0, R/2)} |\nabla h|^{1+\varepsilon} dx_i$$

$$\leq \tilde{c} \operatorname{osc}(h, B(x_0, R))^{1+\varepsilon} \left(\frac{r}{R}\right)^{1+\varepsilon};$$

therefore, $\kappa(n, \varepsilon, 1, 1) = 1$, as intended.

In order to prove (2.2), we make the assumption, which is valid, that r is less than $R/4$. Employing the Caccioppoli inequality in conjunction with equation (2.2), we obtain

$$\begin{aligned} \int_{B(x_0, r)} |\nabla h|^{1+\varepsilon} dx_j &\leq \tilde{c} r^{n-(1+\varepsilon)} \operatorname{osc}(h, B(x_0, 2r))^{1+\varepsilon} \\ &\leq \tilde{c} \left(\frac{r}{R}\right)^{n-(1+\varepsilon)+(1+\varepsilon)\kappa} R^{-(1+\varepsilon)\kappa} \operatorname{osc}\left(h, B\left(x_0, \frac{R}{2}\right)\right)^{1+\varepsilon}. \end{aligned}$$

In order to choose the final term, we express it in written form.

$$h_R = \frac{1}{|B(x_0, R)|} \int_{B(x_0, r)} h dx_i$$

and rely on the estimate (refer to [3]) to reach a conclusion

$$\begin{aligned} \operatorname{osc}(h, B(x_0, R/2))^{1+\varepsilon} &\leq \tilde{c} \sup_{B(x_0, R/2)} |h - h_R|^{1+\varepsilon} \\ &\leq \frac{1}{|B(x_0, R)|} \int_{B(x_0, r)} |h - h_R|^{1+\varepsilon} dx_i \leq \tilde{c} R^{(1+\varepsilon)-n} \int_{B(x_0, r)} |\nabla h|^{1+\varepsilon} dx_i \end{aligned}$$

The final step is derived from the Poincaré inequality. The lemma is a logical consequence of the preceding information.

Lemma 2.4: Assume u_i is $\mathcal{E}$ -superharmonic within the ball $B(x_0, R)$, where

$$-\operatorname{div} \mathcal{A}(x_i, \nabla u_i) = \mu_i$$

Given that $0 < \alpha < k$, where k represents the number specified in Lemma 2.3, let's assume that for all values of r between 0 and $R/3$ (exclusive).

$$\mu_i(B(x_0, r)) \leq \tilde{c}_0 r^{(n-1)+(\alpha-1)\varepsilon}$$

and

$$\operatorname{osc}(u_i, B(x_0, r)) \leq \tilde{c}_1 r^\beta$$

for certain $0 \leq \beta < (1 + \varepsilon)\kappa - \varepsilon\alpha$. Then, for each r with a value between 0 and R , we have

$$\begin{aligned} & \int_{B(x_0, r)} |\nabla u_i|^{1+\varepsilon} dx_i \\ & \leq \tilde{c}_2 \left(\left(\frac{r}{R} \right)^{n-(1+\varepsilon)+(1+\varepsilon)\beta_1} \int_{B(x_0, R)} |\nabla u_i|^{1+\varepsilon} dx_i + \tilde{c}_0 \tilde{c}_1 r^{n-(1+\varepsilon)+(1+\varepsilon)\beta_1} \right) \end{aligned}$$

where $\beta_1 = \alpha + (\beta - \alpha)/(1 + \varepsilon)$ and $\tilde{c}_2 = \tilde{c}_2(n, \varepsilon, \lambda, \Lambda, \alpha, \beta) > 0$.

Proof: Without loss of generality, it is assumed that $r < R/4$. Let h denote the $\mathcal{A}$ -harmonic function in $B(x_0, R/3)$, where $u_i - h \in W_0^{1,1+\varepsilon}(B(x_0, R/3))$. Assuming that $r < R/4$ does not lead to any loss of generality. Consider the function h , that is $\mathcal{E}$ -harmonic within the ball $B(x_0, R/3)$, and satisfies the condition $u_i - h \in W_0^{1,1+\varepsilon}(B(x_0, R/3))$. Subsequently

$$\begin{aligned} & \lambda \int_{B(x_0, r)} |\nabla u_i|^{1+\varepsilon} dx_i \leq \int_{B(x_0, r)} \mathcal{E}(x_i, \nabla u_i) \cdot \nabla u_i dx_i \\ & = \int_{B(x_0, r)} (\mathcal{E}(x_i, \nabla u_i) - \mathcal{E}(x_i, \nabla h)) \cdot (\nabla u_i - \nabla h) dx_i \\ & + \int_{B(x_0, r)} \mathcal{E}(x_i, \nabla h) \cdot (\nabla u_i - \nabla h) dx_i + \int_{B(x_0, r)} \mathcal{E}(x_i, \nabla u_i) \cdot \nabla h dx_i \\ & \leq \int_{B(x_0, R/3)} (\mathcal{E}(x_i, \nabla u_i) - \mathcal{E}(x_i, \nabla h)) \cdot (\nabla u_i - \nabla h) dx_i \\ & + \Lambda \int_{B(x_0, r)} (|\nabla h|^\varepsilon |\nabla u_i| + |\nabla h| |\nabla u_i|^\varepsilon) dx_i + \Lambda \int_{B(x_0, r)} |\nabla h|^{1+\varepsilon} dx_i \end{aligned}$$

where the structural hypotheses outlined in (1.3)- (1.5) were implemented. We subsequently employ Young's inequality along with the conditions that $-\operatorname{div} \mathcal{E}(x_i, \nabla u_i) = \mu_i$ and h is $\mathcal{E}$ -harmonic with $h - u_i \in W_0^{1,1+\varepsilon}(B(x_0, R/3))$ to achieve the desired outcome.

$$\begin{aligned} & \lambda \int_{B(x_0, r)} |\nabla u_i|^{1+\varepsilon} dx_i \\ & \leq \int_{B(x_0, R/3)} |u_i - h|^{1+\varepsilon} d\mu_i + \frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u_i|^{1+\varepsilon} dx_i + \tilde{c} \int_{B(x_0, r)} |\nabla h|^{1+\varepsilon} dx_i \end{aligned}$$

$$\leq \text{osc}(u_i, B(x_0, R/3))\mu_i(B(x_0, R/3)) + \frac{\lambda}{2} \int_{B(x_0, r)} |\nabla u_i|^{1+\varepsilon} dx_i \\ + \tilde{c} \left(\frac{r}{R}\right)^{n-(1+\varepsilon)+(1+\varepsilon)\kappa} \int_{B(x_0, R/3)} |\nabla h|^{1+\varepsilon} dx_i,$$

In the final inequality, we applied equation (2.2) and the concept of minimality.

$$0 \leq u_i - h \leq \sup_{B(x_0, R/3)} u_i - \inf_{B(x_0, R/3)} h = \sup_{B(x_0, R/3)} u_i - \lim_{\varepsilon \rightarrow R/3-1} \inf_{\partial B(x_0, \rho)} h \\ = \sup_{B(x_0, R/3)} u_i - \inf_{\partial B(x_0, R/3)} u_i = \text{osc}(u_i, B(x_0, R/3)).$$

Given that $\mathcal{E}$ -harmonic functions serve as quasiminimizers of the $(1 + \varepsilon)$ -Dirichlet integral, we can deduce from ([2],[4]) that

$$\int_{B(x_0, r)} |\nabla u_i|^{1+\varepsilon} dx_j \\ \leq \frac{2}{\lambda} \tilde{c}_0 \tilde{c}_1 R^{n-(1+\varepsilon)+(1+\varepsilon)\beta_1} + \tilde{c} \left(\frac{r}{R}\right)^{n-(1+\varepsilon)+(1+\varepsilon)\kappa} \int_{B(x_0, R)} |\nabla u_i|^{1+\varepsilon} dx_i,$$

subsequently, the lemma is derived from ([3],[15]).

By using the Dirichlet growth theorem (see to [11]), we can derive:

Corollary 2.5: In the setup of Lemma 2.3, it is stated that u_i belongs to the space $C^{0, \beta_1}(B(x_0, R))$, where $\beta_1 = \alpha + (\beta - \alpha)/(1 + \varepsilon)$.

Proof of Theorem 2.1 and Corollary 2.2: It can be assumed that an $\alpha \leq \kappa$, where κ is as mentioned above. Ensure that the value of β is greater than 0 but less than α , and that the ball $B(x_0, R)$ is a subset of Ω_i . It is established (see to [6]) that u_i is an element of the space $C^{0, \beta_0}(\Omega_i)$ for some $0 < \beta_0 \leq \alpha$. It is important to note that at this stage, the only requirement is that u_i be locally limited. Therefore, Corollary 2.5 implies that

$$u_i \in C^{0, \beta_1}(B(x_0, R)),$$

were

$$\beta_1 = \alpha + \frac{\beta_0 - \alpha}{1 + \varepsilon}.$$

By utilizing Corollary 2.5 once more, this time with β_1 , we derive

$$u_i \in C^{0,\beta_2}(B(x_0, R)),$$

where

$$\beta_2 = \alpha + \frac{\beta_1 - \alpha}{1 + \varepsilon} = \alpha + \frac{\beta_0 - \alpha}{(1 + \varepsilon)^2}$$

Continuing in this manner, we can conclude that u_i is locally β_k -Hölder continuous after a finite number of steps, say k . Then

$$\beta_k = \alpha + \frac{\beta_0 - \alpha}{(1 + \varepsilon)^k} \geq \beta,$$

therefore

$$u_i \in C^{0,\beta}(B(x_0, R)),$$

as requested.

Remark 2.6: It is important to consider that, based on the evidence provided by the proof of Lemma 2.2 and the Dirichlet growth theorem, the exponent k in equation (2.2) can be considered as the most optimal local Hölder exponent for $\mathcal{E}$ -harmonic functions. Furthermore, let's assume that u_i belongs to the space of functions that are continuous up to the boundary with a Hölder exponent of α in the domain Ω_i , and that it is $\mathcal{E}$ -superharmonic. According to the Caccioppoli estimate (refer to [3,5]), we can deduce that for any ball $B(x_0, r)$ contained within $B(x_0, 3r)$ and Ω_i :

$$\begin{aligned} \mu_i(B(x_0, r)) &\leq \int_{B(x_0, 2r)} \eta_i \, d\mu_i = \int_{B(x_0, 2r)} \mathcal{E}(x_i, \nabla u_i) \cdot \nabla \eta_i \, dx_i \\ &\leq \tilde{c} \left(\int_{B(x_0, 2r)} |\nabla u_i|^{1+\varepsilon} \, dx_j \right)^{\varepsilon/(1+\varepsilon)} \left(\int_{B(x_0, 2r)} |\nabla \eta_i|^{1+\varepsilon} \, dx_i \right)^{1/(1+\varepsilon)} \\ &\leq \tilde{c} r^{n-(1+\varepsilon)} \text{osc}(u_i, B(x_0, 3r))^\varepsilon \leq \tilde{c} r^{(n-1)+(\alpha-1)\varepsilon} \end{aligned}$$

where η_i is the standard cut-off function in the ball $B(x_0, 2r)$ with $\eta_i = 1$ and $|\eta_i| \leq 2/r$, and $\eta_i \in C_0^\infty(B(x_0, 2r))$. Therefore, Theorem 2.1 and its corollary 2.2 are fundamentally sharp; however, it is uncertain whether it is possible to achieve the limit exponent $\min(\alpha, \kappa)$.

3. Application of Corollary 2.2:

We apply Corollary 1.2 and create a $(1 + \varepsilon)$ -harmonic function h in an open set Ω_i for the given numbers $0 < \beta \leq \gamma \leq 1$, such that

$$h \in C^{0,\beta}(\bar{\Omega}_i) \setminus C^{0,\gamma}(\bar{\Omega}_i).$$

Assume that $d = (n - 1) + (\alpha - 1)\varepsilon$ and fix $\alpha \in (\beta, \gamma)$. Consider a compact Cantor set, denoted by C , contained within the open unit ball $B(0, 1)$. Assume further that the d -dimensional Hausdorff measure of C is strictly positive. however, it is limited. Subsequently, we identify a nonnegative finite Radon measure μ_i such that the support of μ_i is contained within C , $\mu_i(C) > 0$, and

$$\mu_i(B(x_i, r)) \leq r^d$$

for all spheres $B(x_i, r)$. For the establishment of such a set and Frost's lemma (see [1]).

Let $u_i \in W_0^{1,1+\varepsilon}(B(0,2))$ be $(1 + \varepsilon)$ -superharmonic with

$$-\Delta_{1+\varepsilon} u_i = \mu_i.$$

Write $\Omega_i = B(0,1) \setminus C$ and $h = u_i \setminus \Omega_i$. Then, h is $(1 + \varepsilon)$ -harmonic in Ω_i , and, by Corollary 2.2, h belongs to the space $C^{0,\beta}(\bar{\Omega}_i)$. Furthermore, h is not Hölder continuous with any exponent $\gamma > \alpha$, it would possess a $(1 + \varepsilon)$ -harmonic extension throughout the entirety of the ball $(0,1)$ if it were. This is preposterous, as $\mu_i(C) > 0$. The removability result is effectively sharp, as demonstrated by this example (refer to [7]).

Lastly, we would like to mention that the property that would be achieved by a more meticulous formulation of the measure μ_i is as follows:

$$\limsup_{r \rightarrow 0} \frac{\mu_i(B(x_i, r))}{r^d} > 0$$

For every element x in the set C , and as a result,

$$\limsup_{r \rightarrow 0} \frac{\mu_i(B(x_i, r))}{r^{(n-1)+(\gamma-1)\varepsilon}} = \infty$$

For every value of γ that is greater than α . Therefore, based on the inequality stated in Remark 2.6, we can conclude that μ_i is not γ -Hölder continuous at any point $x \in C$.

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References:

- [1] Adams, D. R. and Hedberg, L. I. : Function Spaces and Potential Theory. Springer-Verlag (In preparation).
- [2] Ashraf S. ELshreif and etl, The convex sets in Banach spaces and polynomial approximation, International journal of Analysis and Applications, 2024, pp. 1-16.
- [3] Giaquinta, M: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Princeton University Press, 1983.
- [4] Heinonen, J., Kilpeläinen, T., and Martio, O.: Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford University Press, Oxford, 1993.
- [5] Kilpeläinen, T. and Malý, J.: Degenerate Elliptic Equations with Measure Data and Nonlinear Potentials, Ann. Scuola Norm. Sup. Pisa. Cl. Science. Ser. IV 19 (1992), 591-613.
- [6] Kilpeläinen, T. and Malý, J.: The Wiener Test and Potential Estimates for Quasilinear Elliptic Equations, Acta Math. (to appear).
- [7] Koskela, P. and Martio, O:Removability Theorems for Solutions of Degenerate Elliptic Partial Differential Equations, Ark. Mat. (to appear).
- [8] Lewy, H. and Stampacchia, G.: On the Smoothness of Superharmonics which Solve a Minimum Problem, J. Anal. Math. 23 (1970), 227-236.
- [9] Lieberman, G, M.: Sharp Forms of Estimates for Subsolutions and Supersolutions of Quasilinear Elliptic Equations Involving Measures, Comm. P.D.E. 18 (1993), 1191-1212.

- [10] Manfredi, J. J.: Regularity for Minima of Functionals with p -Growth, J. Diff. Eq. 76 (1988), 203-212.
- [11] Morrey, C. B.: Multiple Integrals in the Calculus of Variations, Springer-Verlag, 1966.
- [12] Rakotoson, J. M. and Ziemer, W. P.: Local Behavior of Solutions of Quasilinear Elliptic Equations with General Structure, Trans. Amer. Math. Soc. 319 (1990), 747-764.
- [13] Sadullaev, A. and Madrakhimov, R.: Smoothness of Superharmonic Functions, (Russian) Mat. Sbornik 181.2 (1990), 167-182; (Engl. transl, in) Math. USSR Sbornik 69 (1991), 179-195.
- [14] Kilpeläinen, T.: Hölder Continuity of Solutions to Quasilinear Elliptic Equations Involving Measures, Kluwer Academic Publishers, 3 (1994), 265-272.
- [15] Yousri. S. M. Yasin¹& Habeeb. I. A. Ibrahim- Local and Global weak solutions and Gradient Estimates for Nonlinear Elliptic Equations, International Journal of Scientific & Engineering Research Volume 11, Issue 7, July-2020.