

FULL PAPER

Continuous Sequentially Differentiable Functions in Banach Spaces

Abstract

Prepared by

Habeeb Ibrahim–
Department of Mathematics
College of Science
Qassim University
Buraydah 51452 -Saudi Arabia
Ha.Ibrahim@qu.edu.sa

. We gave the concepts of continuous differentiable functions that are weakly q -sequentially, and uniformly q -convergent sets. Next, we derived a sufficient condition for Banach spaces that possess the q -Schur property or do not include a copy of ℓ_1 . Finally, we proved that, if the differentiable function $g'(Y)$ is relatively compact, and weakly q -sequentially continuous function, then $g'(a_n)$ converges to $g'(a)$.

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1. Introduction:

Consider the two Banach spaces A and B , where $A, B \subset \mathbb{R}$ and $V \subset A$ is considered to be an open set. Let $\text{bnd} V$, $\text{bdd} V$ and (B, S) denote the boundary of V , bounded of V and Banach space respectively. If Y is bounded, and the $d(Y, \text{bnd} V) > 0$, then the set Y is called V -bounded set [11]. Suppose that $C^{(1+\alpha)V}(V, B)$ signifies the space occupied by all differentiable functions $g: V \rightarrow B$, where its derivative $g': V \rightarrow L(A, B)$ is uniformly

continuous on the V -bdd subsets of V . The properties of the space $C^{(1+\alpha)V}$ functions between (B.S)s have been examined by a lot of authors in recent years. For additional details in this field, we direct the reader to [14, 15, 20] and their citations. Consider a function $g : V \rightarrow B$. If g transforms the V -bdd subsets of V into subsets of the space B that are substantially compact, then g is compact.

According to Aron ([3, Theorem 1.7]), a holomorphic linear function $g : A \rightarrow C$ is compact on bounded subsets if and only if its derivative $g' : A \rightarrow A^*$ is a compact function. Bombal et al. [1, 6], proved that, if g is a holomorphic bounded function on a complex (B.S) A , then the derivative $g' : A \rightarrow A^*$ accepts bounded sets into specific families of sets if and only if g can be factored as $g = f \circ S$, where f is a holomorphic function of bounded type and S is in some related operator ideal. Thereafter, Cilia et al. [12] demonstrated the uniformly weakly continuity of the holomorphic bounded function g and the compactness of g' . In (B.S)s, an operator $T : A \rightarrow B$ of linear bounded type is considered completely continuous if and only if $T^* : B^* \rightarrow A^*$ where A^* is bounded set, and $T^*(B^*)$ is uniformly continuous and complete subset of A^* [18].

As we remember from [23], a subset $K \subseteq L(A, B)$ is said to be uniformly completely continuous if the following condition hold:

$$\lim_{k \rightarrow \infty} \sup_{T \in K} \sum_{k=1}^{\infty} \|T(a_k)\| = 0, \forall (a_k)_k \in A,$$

where $(a_k)_k$ is a weakly null sequence.

This study inspired Cilia and Gutiérrez to provide a similar result for differentiable maps ([10, Theorem 2.1]). Indeed, if $g \in C^{(1+\alpha)V}(V, B)$, they demonstrated that g transforms the V -bdd sequences of weakly Cauchy type into convergent sequences if and only if g' transforms the V -bdd Rosenthal subsets of V into subsets of A^* that are uniformly completely continuous. Li and colleagues [21] recently extended the notions of (L) sets and (U) sets to the q -(U) sets for $1 \leq q < \infty$. In (B.S)s, a bounded linear operator $T : A \rightarrow B$ is said to be q -convergent if and only if the adjoint $T^* : B^* \rightarrow A^*$ converts the bounded subsets of B^* into q -(V) $\subset A^*$ [17]. Motivated by the aforementioned research, we present the concepts of weakly continuous functions of q -sequentially type and uniformly q -convergent sets. We gave a sufficient condition for (B.S)s that possess the q -Schur property or do not include a copy of ℓ_1 . Lastly, we demonstrate that if $V \subset A$ is open and convex set and $f \in C^{(1+\alpha)V}(V, B)$, then g produces norm convergent sequences from V -bdd and weakly q -Cauchy sequences if and only if g' produces uniformly q -convergent subsets of $L(A, B)$ from V -bdd sets and weakly q -precompact sets which are subset of V .

2. Basic Definitions:

In this work, $1 \leq q < \infty$ and $1 \leq q < p \leq \infty$, unless for the cases of other assumptions. Additionally, we assume that $V \subseteq Y$ signify to an open convex subset, A and B represent real (B.S)s, and q^* is the Hölder conjugate of q . "Operator" always refers to a bounded linear operator. The bounded linear functionals' dual space on any Banach space A will be represented by A^* . Additionally, we use A_x to represent the closed unit ball in A . The adjoint of a bounded linear operator $T : A \rightarrow B$ is represented by T^* . $L(A, B)$ and $W(A, B)$, respectively, shall stand for the all-weakly compact operators' space, and all bounded

operators of linear type from A to B . The space all functions of k -linear type from $A_1, \dots, A_k$ into B is represented by $L^k(A_1, \dots, A_k, B)$ if $A_1, \dots, A_k$ are (B.S) s. The existence of a k -linear function $B: A \times \dots \times A \rightarrow B$ such that $P(a) = B(a, \dots, a)$ for every $a \in A$ indicates that the function $P: A \rightarrow B$ is a k -homogeneous (continuous) polynomial. For every k -homogeneous continuous polynomial from A into B , the space is represented as $P(^kA, B)$. The symmetric k -linear function $\hat{P}: A \times \dots \times A \rightarrow B$ can be associated with each $P \in P(^kA, B)$ in such a way that $P(x) = \hat{P}(a, \dots, a)$. A mapping $g: A \rightarrow B$ is holomorphic if there is $(P_k)_{k=0}^\infty$ of polynomials where $P_k \in P(^kA, B)$ and $r > 0$ for each $a \in A$. Moreover, the Taylor series of g around x may yields:

$$g(b) = \sum_{k=0}^{\infty} P_k(b - a)$$

consistently for $b - a < r$. For the space of all holomorphic functions from A into B , we employ the notation $\mathcal{H}(A, B)$. When g is bounded on bounded sets, the mapping $f \in \mathcal{H}(A, B)$ is referred to as bounded type. $\mathcal{H}_a(A, B)$ represents the space that includes all bounded functions from A to B . The real field $\mathbb{R}$ is interpreted as the range space if it is not included in the aforementioned notations, for example, $\mathcal{H}(A) = \mathcal{H}(A, \mathbb{R})$, $P(^kA) = P(^kA, \mathbb{R})$. Moreover, $\mathcal{H}_a(A) = \mathcal{H}_a(A, \mathbb{R})$. We write $I(a, b)$ for the segment with boundaries a and b given $a, b \in A$, which means:

$$I(a, b) = \{a + \lambda(b - a) : 0 \leq \lambda \leq 1\}.$$

Definition 2.1: If an operator converts weakly convergent sequences into norm convergent sequences, it is said to be entirely continuous. The symbol $CC(A, B)$ represents the subspace of all such operators. From [16, 25], we recall that the $(a_n)_n$ in A is said to be weakly q -summable if $(a^*(a_n))_n \in \ell_q$ for every $a^* \in A^*$. If the sequence $(a_n)_n$ is weakly q -summable, then $(a_n)_n$ in A is weakly q -convergent to $a \in A$. Let $\varphi_k = (n_k)_k$ be an increasing sequence. If the sequence $(a_n - a)_n$ is weakly q -summable for any (φ_k) , then $(a_n)_n$ in A is weakly q -Cauchy [9]. The weakly ∞ -Cauchy sequences are exactly the weakly Cauchy sequences, and it should be noted that any weakly q -convergent sequences are also weakly q -Cauchy.

Definition 2.2: If an operator $T: A \rightarrow B$ converts weakly q -summable sequences into norm null sequences, it is said to be q -convergent [1, 7]. For every q -convergent operator from A into B , the class is represented by $C_p(A, B)$. If a (B.S) A 's identity operator is q -convergent, then A possesses the q -Schur property. A bounded set $K \subset X^*$ is called q -(U), if for each sequence $(a_n)_n \in A$ that is weakly and q -summable, the following holds:

$$\lim_i \sup_{a^* \in K} \sum_{i=1}^n \|a^*(a_i)\| = 0$$

Definition 2.3: A set $K \subset A$ is called weakly q -precompact, if each $(a_n)_n \in K$ there is $(a_{n_k})_k \in K$ which is weakly q -Cauchy type [9].

It should be noted that the ∞ -precompact sets that are weakly type are exactly the Rosenthal sets, or weakly precompact sets. For unclear terms, we direct the reader to the traditional sources [2, 16].

3. Main results:

We give description for continuous functions that are weakly q -sequentially type in this section. Specifically, we demonstrate that, $g: V \rightarrow B$ is a continuous weakly q -sequentially if and only if g' converts the V -bdd and the weakly subsets of V that are q -precompact into subsets into uniformly subsets of $L(A, B)$ that are q -convergent.

Definition 3.1. Suppose that $K \subset L(A, B)$. K is said to be a uniformly set of q -convergent type, if for each $(a_n)_n \in A$, that is weakly q -summable, the following holds:

$$\limsup_n \sum_{T \in K} \sum_{n=1}^{\infty} \|T(a_n)\| = 0.$$

Directly, from Definition 3.1, we have the following basic results:

Proposition 3.2:

- (a) Every $K \subset L(A, B)$ is uniformly and q -convergent set.
- (b) A hull that is absolutely closed convex of uniformly q -convergent set in $L(A, B)$ is uniformly q -convergent.
- (c) Let $K_1, \dots, K_n$ be uniformly q -convergent sets in the space $L(A, B)$; then the finite union of the k_i 's and the summing of k_i ' are uniformly sets of q -convergent type in the space $L(A, B)$.

Remark 3.3:

- (a) The notion of uniformly sets of q -convergent type and the definition of q -(U) sets coincide if $K \subset A^*$. Additionally, whenever $1 \leq q < p < \infty$ any set $K \subset L(A, B)$ that is uniformly p -convergent is uniformly q -convergent. Finding the circumstances in which any subset in $L(A, B)$ that is consistently q -convergent is uniformly p -convergent too.

Finding the circumstances in which any subset of $L(A, B)$ that is consistently q -convergent set is uniformly p -convergent. would also be intriguing. We give description for the sets in (B.S) s that are uniformly and q -convergent in $A^*[5]$.

- (b) Let $P \in P(^k A)$. The equivalence (a) $\Leftrightarrow$ (e) in ([6, Corollary 9]) can be easily verified. It indicates that conjugate of the operator P from A to $P(^k A)$ given by $\bar{P}(a)(b) = \hat{P}(a, \dots, a, b) \in C_p$ where $(a, b) \in A$ if and only if $P' \in P(^k A, A^*)$ defined by $P'(a)(b) = k \hat{P}(a, \dots, a, b)$ takes the bounded set of A into q -convergent sets that are uniformly in A^* .

- (c) The bounded holomorphic function g on a complex (B. S) A is known to take the bounded sets into q -convergent sets that are uniformly in A^* through its derivative $g' : A \rightarrow A^*$ if and only if g can be factored in the form $g = f \circ S$, where S is q -convergent and $f \in \mathcal{H}_a(A)$. The example that follows demonstrates that $L(\ell_2, B)$ is not consistently p -convergent but does have a q -convergen subset of uniformly type.

Example 3.4: Suppose that B is an arbitrary Banach space and $A = \ell_2$. Since the space ℓ_2 does not possess 2-Schur property, then A_{ℓ_2} is not a 2-(U) set in the space ℓ_2 and

consequently, A_{ℓ_2} is not a uniformly 2-convergent subset of ℓ_2 . However, there isn't a copy of c_0 in the space ℓ_2 . Consequently, ℓ_2 possesses the 1-Schur characteristic [13]. Therefore, A_{ℓ_2} is a subset in ℓ_2 that is uniformly 1-convergent, and A_{ℓ_2} is a 1-(U) set. Suppose that $S: \mathbb{R} \rightarrow X$ denotes the operator provided by $S(\lambda) := \lambda x_0$ ($\lambda \in \mathbb{R}$), and that $0 \neq x_0 \in A_X$. Let us define an operator T from the space ℓ_2 to the space $K(\ell_2, B)$ by $T(\psi)(h) := \psi(h)x_0$: $\psi \in \ell_2^* = \ell_2$, $h \in \ell_2$. Thus

$T(\psi)(h) := \psi(h)x_0$, for $\psi \in \ell_2^* = \ell_2$ and $h \in \ell_2$. And we have,

$$\|T(\psi)\| = \sup_{h \in A_{\ell_2}} \|T(\psi)(h)\| = \sup_{h \in A_{\ell_2}} \|\psi(h)x_0\| = \|\psi\|.$$

There is a subset in $K(\ell_2, B)$ of uniformly 1-convergent type that is not uniformly 2-convergent because uniformly q -convergent sets are stable under isometry. Note that if the set $\{y_n : n \in \mathbb{N}\}$ is V -bdd, then a sequence $(a_n) \subset V$ is also V -bdd. If the function $g: V \rightarrow X$ converts the weakly Cauchy sequences and the V -bdd and of V into norm of the convergent sequences in B , then it is weakly sequentially continuous. We use the symbol $C_{wsc}(V, B)$ to denotes the all such function space in B [10].

Definition 3.5: Assume that V signify to a convex set that is an open subset of A and $1 \leq q \leq \infty$. If the function $g: V \rightarrow B$ transforms weakly q -Cauchy and V -bdd sequences of V into the norm of convergent sequences in B , we call it a weakly q -sequentially continuous map. And the symbol $C_{wsc}^q(V, B)$ represents the space of all such functions.

Remark 3.6: The continuous maps that are weakly ∞ - sequentially easily shown to be weakly sequentially continuous. Additionally, it is evident that $C_{wsc}^p(V, B) \subseteq C_{wsc}^q(V, B)$ for every A, B , and for every open subset $V \subseteq A$, when $1 \leq q < p \leq \infty$. Nevertheless, we need an example of a mapping $g \in C^{(1+\alpha)v}(V, B) \cap C_{wsc}^q(V, B)$ that is not a part of $C_{wsc}^p(V, B)$. It would also be interesting to find the conditions under which the V -bdd and the sequences that are weakly q -Cauchy are transformed into the norm of convergent sequences by any mapping in the space $C^{(1+\alpha)v}(V, B) \cap C_{wsc}^q(V, B)$.

Proposition 3.7: Suppose that V signifies a convex set that is an open subset of A , and $g \in C^{(1+\alpha)v}(V, B)$. If $g' \in C_{wsc}^q(V, B)$, then $g \in C_{wsc}^q(V, B)$.

Proof. Suppose that $(a_n)_n$ denotes a V -bdd weakly q -Cauchy sequence. By using Theorem 6.4 of ([8,]), we have:

$$\sum_{n,m=1}^{\infty} \|g(a_n) - g(a_m)\| \leq \sum_{n,m=1}^{\infty} \|g'(c_{n,m})(a_n - a_m)\|,$$

for some $c_{n,m}$ in $I(a_n, a_m)$. According to the hypothesis, the norm of a sequence $(g'(c_{n,m}))$ converges to some $T \in C_q(A, B)$, since $(c_{n,m})$ is V -bdd and weakly q -Cauchy sequence. Consequently, we have,

$$\lim_{n,m \rightarrow \infty} \sum_{n,m=1}^{\infty} \|g'(c_{n,m})(a_n - a_m) - T(a_n - a_m)\| + \sum_{n,m=1}^{\infty} T(a_n - a_m).$$

$$\leq \lim_{n,m \rightarrow \infty} \sum_{n,m=1}^{\infty} \|g'(c_{n,m})(a_n - a_m) - T(a_n - a_m)\|$$

$$+ \lim_{n,m \rightarrow \infty} \sum_{n,m=1}^{\infty} \|T(a_n - a_m)\| = 0$$

Hence,

$$\lim_{n,m \rightarrow \infty} \sum_{n,m=1}^{\infty} \|g'(c_{n,m})(a_n - a_m)\| = 0$$

Lemma 3.8. Assume that A does not possess the q -Schur property, then there is a non p -convergent operator from A to the space $L_{\infty}[0,1]$. Assume that A lacks the q -Schur property, the operator $T: A \rightarrow K_{\infty}[0,1]$. is non- q -convergent [8].

Proof: According to ([2, Theorem 2.5.7]), A embeds isometrically into the space ℓ_{∞} if A is separable. As a result, an isometry $T: A \rightarrow \ell_{\infty}$ exists, which is clearly not q -convergent. Without using the q -Schur property, it is simple to demonstrate that A has a separable subspace C if it is not separable. As a result, there is an isometry map T_1 in the space $L(C, \ell_{\infty})$. Known that ℓ_{∞} is injective, then we can determine an operator $T: A \rightarrow \ell_{\infty}$, that is linear bounded which is and extending of the operator T_1 .

T is obviously not a q -convergent operator. Since the space ℓ_{∞} is isomorphic to the space $L_{\infty}[0,1]$, then we can obtain an operator $T: A \rightarrow L_{\infty}[0,1]$ that is non q -convergent [2]. With a few small adjustments, the proof of the following lemma is comparable to the proof of ([4, Theorem 2.3]). Therefore, we do not include its proof.

Lemma 3.9: Only when a polynomial is q -convergent does it become weakly q -sequentially continuous. In short, if the inclusion $W(A, B) \subseteq C_q(A, B)$ holds for all (B.S)s B , then A possesses the Dunford-Pettis-property of order p [7, 19].

Theorem 3.10: Let V be a subset of A that is open and convex. If the derivative $P' \in P(^{k-1}A, L(A, B))$ is weakly q -sequentially continuous for all k in $\mathbb{N}$ where $k \geq 2$, every (B.S) Y , and every weakly polynomial $P \in P(^kA, B)$ that is q -sequentially continuous, then either A satisfies the q -Schur condition or A contains no copy of the space ℓ_1 .

Proof: Contrary, suppose that A lacks the q -Schur property and has a copy of ℓ_1 . Thus, a q -convergent operator from A onto ℓ_2 exists [16]. We find a surjective operator $S \in C_p(A, L_2[0,1])$, that is because ℓ_2 is isometrically isomorphic to the space $L_2[0,1]$. Conversely, according to Lemma 3.1, the operator T is non- q -convergent $T: A \rightarrow L_{\infty}[0,1]$.

Now, let $k \in \mathbb{Z}: k \geq 2$. We define a k -linear function $L: A^k \rightarrow L_k[0,1]$ by:

$$L(a_1, \dots, a_k) = \frac{1}{k} [S(a_1)T(a_2) \cdots T(a_k) + \cdots + T(a_1)T(a_2) \cdots T(a_{k-1})S(a_k)]$$

for $a_1, \dots, a_k \in A$. It is evident that L is continuous, symmetric, and well defined. Consider the related polynomial $P(a) = S(a)T(a)^{k-1}$ for any $a \in A$. Let P be weakly continuous in the q -sequence, and the sequence $(a_n - a)_n \in Y_A$ is weakly q -summable. So, we have:

$$\begin{aligned}
& \sum_{n=1}^{\infty} \|P(a_n) - P(a)\| = \sum_{n=1}^{\infty} \|S(a_n)T(a_n)^{k-1} - S(a)T(a)^{k-1}\| \\
& \leq \sum_{n=1}^{\infty} \|S(a_n)T(a_n)^{k-1} - S(a)T(a_n)^{k-1}\| + \|S(a)T(a_n)^{k-1} - S(a)T(a)^{k-1}\| \\
& = \left(\sum_{n=1}^{\infty} \int_0^1 |S(a_n)(t) - S(a)(t)|^2 |T(a_n)(t)^{2k-2}| dt \right)^{\frac{1}{2}} \\
& \quad + \left(\sum_{n=1}^{\infty} \int_0^1 |S(a)(t)|^2 |T(a_n)(t)^{k-1} - T(a)(t)^{k-1}|^2 dt \right)^{\frac{1}{2}} \\
& \leq \|T\|^{k-1} \left(\sum_{n=1}^{\infty} \int_0^1 |S(a_n) - S(a)(t)|^2 dt \right) + \langle S(a)^2, \sum_{n=1}^{\infty} [T(a_n)(t)^{k-1} - T(a)(t)^{k-1}]^2 \rangle^{\frac{1}{2}} \\
& = \|T\|^{k-1} \sum_{n=1}^{\infty} \|S(a_n - a)\|_{L_2} + \langle S(a)^2, \sum_{n=1}^{\infty} [T(a_n)(t)^{k-1} - T(a)(t)^{k-1}]^2 \rangle^{\frac{1}{2}}
\end{aligned}$$

Given that $S \in C_p(A, L_2[0,1])$, $\lim_n \sum_{n=1}^{\infty} \|S(a_n - a)\|_{L_2} = 0$. Let $\psi(f) := f^m$ defines the polynomial ψ from $L_{\infty}[0,1]$ to itself be taken into consideration for $m \in \mathbb{N}$, where $(m \geq 2)$. The sequence $\sum_{n=1}^{\infty} ([T(a_n)^{k-1} - T(a)^{k-1}]^2)_n$ is null and weakly in the space $L_{\infty}[0,1]$ since according to the Dunford-Pettis property in $L_{\infty}[0,1]$, weakly q -convergent sequences are transformed into norm convergent sequences by this polynomial. Alternatively, $S(a)^2 \in L_1[0,1]$. Thus

$$\lim_{n \rightarrow \infty} \langle S(a)^2, \sum_{n=1}^{\infty} [T(a_n)(t)^{k-1} - T(a)(t)^{k-1}]^2 \rangle^{\frac{1}{2}} = 0$$

Consequently, $\lim_n \sum_{n=1}^{\infty} \|P(a_n) - P(a)\| = 0$. Thus, P is the weakly continuous polynomial of q -sequentially type, as implied by Lemma 3.9. P' is not weakly continuous q -sequentially type, which is contradictory, according to the same reasoning as in the proof of ([10, Theorem 2.9]). The following identical result is obtained by applying the same logic as in the proof of ([10, Theorem 2.1]).

Theorem 3.10. suppose that $g \in C^{(1+\alpha)v}(V, B)$ and let $V \subseteq A$ be an open convex subset. Then, the following claims are interchangeable:

- (a) $g \in C_{wsc}^p(V, B)$;
- (b) The sequence $(g'(a_n)(h_n))_n$ norm converges in B for all V -bounded and weakly q -Cauchy sequences (a_n) and weakly p -Cauchy sequences $(h_n) \subset A$;
- (c) Every weakly q -summable sequence $(h_n)_n \subset A$ and every V -bounded weakly q -Cauchy sequence $(a_n)_n$ yield the following:

$$\lim_{n \rightarrow \infty} \sup_m \sum_{m=1}^{\infty} \|f'(a_m)(h_n)\| = 0$$

(d) Every weakly q -summable sequence $(h_n)_n \subset A$ and every V -bounded weakly q -Cauchy sequence $(a_n)_n$ have:

$$\lim_n g'(a_n)(h_n) = 0$$

(e) g' converts V -bdd and weakly subsets of V that is q -precompact into subsets of the space $L(A, B)$ that are uniformly p -convergent.

Proof. (a) $\Rightarrow$ (b) Assume that $(a_n)_n$ is V -bdd sequence of weakly q -Cauchy type. Consider a weakly sequence in A that is q -Cauchy, $(h_n)_n$ is assumed to be bounded without loss of generality. Let $d := \min\{1, \text{dist}(Y, \partial V)\}$. and use $Y := \{a_n : n \in \mathbb{N}\}$. It is simple to demonstrate that the set

$$Y' := Y + \frac{d}{2} Y_A \subset V$$

is V -bounded as well. The function g' is uniformly continuous on Y' since $g \in C^{(1+\alpha)_V}(V, B)$. Therefore, if $t_1, t_2 \in Y'$ satisfy $\sum_{n=1}^{\infty} \|t_n - t_{n+1}\| < 2\delta$, then

$$\sum_{n=1}^{\infty} \|g'(t_n) - g'(t_{n+1})\| < \frac{\varepsilon}{4} \quad (3.1)$$

Let $c \in I(a_n, a_n + \delta h_n)$ for some natural numbers n in $\mathbb{N}$, then

$$\sum_{n=1}^{\infty} \|c - a_n\| \leq \delta \|h_n\| < \delta < 2\delta < \frac{d}{2}$$

and so,

$$c = a_n + (c - a_n) \in Y' = Y + \frac{d}{2} Y_A$$

The Mean Value Theorem ([8, Theorem 6.4]) and formula (3.1) immediately lead to the following:

$$\begin{aligned} & \sum_{n=1}^{\infty} \|g'(a_n)(\delta h_n) - g(a_n + \delta a_n) + g(a_n)\| \\ & \leq \sup_{c \in K} \sum_{n=1}^{\infty} \|g'(c) - g'(a_n)\| \|\delta h_n\| \leq \frac{\delta \varepsilon}{4} : K = I(a_n, a_n + \delta h_n) \end{aligned}$$

Likewise,

$$\begin{aligned} & \sum_{m=1}^{\infty} \|g(a_m + \delta h_m) - g(a_m) - g'(a_m)(h_m)\| \\ & \leq \sup_{c \in K} \sum_{m=1}^{\infty} \|g'(c) - g'(a_m)\| \|\delta h_m\| \leq \frac{\delta \varepsilon}{4} : K = I(a_m, a_m + \delta h_m). \end{aligned}$$

In other words, the two sequences $(a_n + \delta h_n)_n$ and $(a_n)_n$ are weakly q -Cauchy in V and V -bounded. The two sequences $(g(a_n + \delta h_n))_n$ and $(g(a_n))_n$ are therefore norm convergent in B according to the hypothesis. For $n, m > n_0$, we can determine $n_0 \in \mathbb{N}$ such that:

$$\sum_{n=1}^{\infty} \|g(a_n + \delta h_n) - g(a_m + \delta h_m)\| < \frac{\varepsilon \delta}{4}, \quad \sum_{m=1}^{\infty} \|g(a_n) - g(a_m)\| < \frac{\varepsilon \delta}{4}$$

Therefore, if $n, m > n_0$, we have

$$\sum_{n,m=1}^{\infty} \|g'(a_n)(h_n) - g'(a_m)(h_m)\| < \varepsilon$$

(b) $\Rightarrow$ (c) Consider a V bdd sequence $(a_n)_n$ that is weakly Cauchy, and a weakly sequence $(h_n)_n$ that is q -summable in A . Using part (b), for every $h \in A$, the sequence $g'(a_n)(h)$ is bounded in B . However, there exists a subsequence $(a_{m_k})_k$ of $(a_m)_m$ in V , such that:

$$\sum_{k=1}^{\infty} \|g'(a_{m_k})(h_k)\| \geq \sup_m \sum_{k=1}^{\infty} \|g'(a_{m_k})(h_k)\| - \frac{1}{k}, (k \in \mathbb{N}).$$

Given the weak q -Cauchy nature of the sub sequences $(x_{m_k})_k \in V$ and, a weakly q -Cauchy $(h_1, 0, h_2, 0, h_3, 0, \dots) \in A$, then the sequence:

$$(f'(x_{m_1})(h_1), 0, f'(x_{m_2})(h_2), 0, f'(x_{m_3})(h_3), 0, \dots)$$

converges in B . Since $\lim_k \sum_{k=1}^{\infty} g'(a_{m_k})(h_k) = 0$, from part (b) we have:

$$\lim_k \sup_m \sum_{m,k=1}^{\infty} \|g'(a_m)(h_k)\| = 0$$

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (e): Suppose K be a V -bounded set that is weakly q -precompact. The set $g'(K)(h)$ is obviously confined in B for each $h \in A$. Let $(h_n)_n$ be a sequence in A that is weakly q -summable. If $(h_{n_k})_k \subset (h_n)_n$, then for each k in $\mathbb{N}$, there is $x_k \in K$ such that:

$$\sup_{a \in M} \sum_{M=1}^{\infty} \|g'(x)(h_{n_M})\| \leq \|g'(x_M)(h_{n_M})\| + \frac{1}{M}.$$

Given that M is a set of weakly q -precompact type, the sequence $(x_k)_k$ permits a subsequence $(x_{k_r})_r$ of weakly q -Cauchy type. Thus, according to the theory,

$$\lim_r \sum_{r=1}^{\infty} \|f'(a_{k_r})(h_{n_{k_r}})\| = 0$$

From this, we have $\lim_r \sup_{a \in M} \sum_{r=1}^{\infty} \|g'(x)(h_{n_{M_r}})\| = 0$. As a result, each subsequence of $(\sup_{a \in M} \sum_{n=1}^{\infty} \|f'(a)(h_n)\|)_n$ has a subsequence at zero. Consequently, $\lim_n \sup_{a \in M} \sum_{n=1}^{\infty} \|g'(x)(h_n)\| = 0$, that is because the sequence itself converges to 0.

(e) $\Rightarrow$ (a) Assume that $(a_n)_n$ be a weakly V -bdd sequence of q -Cauchy type. For all n, m in $\mathbb{N}$, the segment $I(a_n, a_m)$ is included in V since V is convex. Recall theorem 6.4 of [8] we find $c_{n,m} \in I(a_n, a_m)$ such that:

$$\limsup_{n,m} \sum_{i,j \in \mathbb{N}}^{\infty} \|g'(c_{i,j})(a_n - a_m)\| = 0$$

Therefore, $\lim_{n,m \rightarrow \infty} \sum_{n,m=1}^{\infty} \|g(a_n) - g(a_m)\| = 0$

Remark 3.11: Let $V \subseteq A$ signifies an open subset, and $g \in C^{(1+\alpha)u}(V, B) \cap C_{wsc}^q(V, B)$. Then, by part (b) of Theorem 3.3, we have $g'(V) \subseteq C_q(A, B)$.

Theorem 3.12: Assume that $V \subset A$ be an open set, Y_A is weakly q -precompact, and $g \in C^{(1+\alpha)v}(V, \mathbb{R}) \cap C_{wsc}^q(V, \mathbb{R})$. The function g' is compact with values in $C_q(A, \mathbb{R})$. if all sets that of uniformly q -convergent type in $K(A, \mathbb{R})$ are uniformly totally continuous.

Proof: According to Remark 3.11, we obtain that $C_q(A, \mathbb{R}) = K(A, \mathbb{R})$ contains the range of g' , where we apply the weakly q -precompact nature of Y_A [9]. Assume that Y is a V -bounded set. Considering h to be in A , let $(a_n)_n \subset Y$ be a sequence. The assumption that $(a_n)_n$ is weakly q -Cauchy is supported by the weak q -precompactness of Y_A . So, according to Theorem 3.12 part (v), $g'(Y)(h)$ is reasonably compact in $\mathbb{R}$ since the sequence $(g'(a_n)(h))_n$ converges in $\mathbb{R}$. Additionally, according to Theorem 3.11(v), $g'(Y)$ is a set of uniformly q -convergent type in the space $K(A, \mathbb{R})$. Accordingly, $g'(Y)$ is uniformly totally continuous in $K(A, \mathbb{R})$ according to the hypothesis. However, we note that A does not include a copy of ℓ_1 by utilizing the fact that Y_A is weakly and q -precompact once more. Consequently, it is implied by ([22, Theorem 1]) that $g'(Y)$ is reasonably compact in $C_q(A, \mathbb{R})$.

Definition 3.13: A mapping $g: V \rightarrow B$ is said to be q -convergent if the sequence $(g(a_n))_n$ norm converges to $g(a)$ for each V -bounded and weakly q -converging sequence $(a_n)_n$ to $a \in V$. We use $C_{Cq}(A, B)$ to represent the space of q -convergent mappings. It is evident that we have $C_{wsc}^q(V, B) \subseteq C_{Cq}(V, B)$ for every A, B and for each open subset $V \subseteq A$. We have no example of a mapping that does not belong to $C_{wsc}^q(V, B)$ such that $g \in C^{(1+\alpha)v}(V, B) \cap C_{Cq}(V, B)$ [25].

Theorem 3.14. Let $g \in C^{(1+\alpha)v}(V, B)$ and let V denotes an open and convex subset of A . The following propositions are true if g' converts the V -bdd subsets of V that are weakly q -compact into comparatively compact subsets of the space $K(A, B)$:

(a) $g \in C_{Cq}(V, B)$;

(b) $g' \in C_{Cq}(V, K(A, B))$.

Proof: (a) Consider the sequence $(a_n)_n$ to be V -bounded and weakly q -convergent to $a \in V$. For any n in $\mathbb{N}$, there exists $c_n \in I(a_n, a)$ such that, according to the Mean Value Theorem ([8, 24]), the set $M := \{a_n: n \in \mathbb{N}\} \cup \{a\}$ is a weakly q -compact subset of V , such that:

$$\sum_{n=1}^{\infty} \|g(a_n) - g(a)\| \leq \sum_{n=1}^{\infty} \|g'(c_n)(a_n - a)\|$$

$$\leq \sup_k \sum_{n=1}^{\infty} \|f'(c_k)(a_n - a)\| : n \in \mathbb{N}$$

Assume that the sequence $(a_n)_n$ is weakly q -convergent and V -bounded. Suppose that, $Y := \{a_n : n \in \mathbb{N}\} \cup \{a\}$ and consider the distance $d := \min\{1, \text{dist}(Y, \partial V)\}$. The set:

$$Y' := Y + \frac{d}{2} Y_A \subset V$$

is V -bounded as well. The map g' is uniformly continuous on Y' because $g \in C^{(1+\alpha)V}(V, B)$. Then, for $\varepsilon > 0$, there is $0 < \delta < \frac{d}{4}$ satisfies $\sum_{n=1}^{\infty} \|t_n - t_{n+1}\| < 2\delta$ for $t_1, t_2 \in B'$. And then

$$\sum_{n=1}^{\infty} \|g'(t_n) - g'(t_{n+1})\| < \frac{\varepsilon}{4} \quad (3.2)$$

Let $n \in \mathbb{N}$ and $h \in Y_A$. If $c \in I(a_n, a_n + a_n h)$, then we have

$$\sum_{n=1}^{\infty} \|c - a_n\| \leq 2\delta < \frac{d}{2},$$

and so, $c = a_n + (c - a_n) \in Y' := Y + \frac{d}{2} Y_A$. Thus, $I(a_n, a_n + \delta h) \subset Y'$.

Likewise, $I(a, a + \delta h) \subset Y'$. From equation (3.2) and the Mean Value Theorem ([8, Theorem 6.4]), we immediately get:

$$\begin{aligned} & \sum_{n=1}^{\infty} \|g'(a_n)(\delta h) - g(a_n + \delta h) + g(a_n)\| \\ & \leq \sup_{c \in K} \sum_{n=1}^{\infty} \|g'(c) - g'(a_n)\| \|\delta h\| \leq \frac{\delta \varepsilon}{4} \end{aligned}$$

where $K = I(a_n, a_n + \delta h)$.

Likewise,

$$\begin{aligned} & \sum_{n=1}^{\infty} \|g(a + \delta h) - g(a) - g'(a)(\delta h)\| \\ & \leq \sup_{c \in K} \sum_{n=1}^{\infty} \|g'(c) - g'(a)\| \|\delta h\| \leq \frac{\delta \varepsilon}{4} \end{aligned}$$

where $K = I(a_n, a_n + \delta h)$.

In other words, the two sequences $(a_n + \delta h)_n$ and $(a_n)_n$ are V -bdd and weakly q -convergent in V to the two limits $a + \delta h$ and a , respectively. Thus, for a positive integer $n > n_0$, we have:

$$\sum_{n=1}^{\infty} \|g'(a_n)(h) - g'(a)(h)\| < \varepsilon.$$

Thus, for each $h \in A$ we have:

$$\lim_n \sum_{n=1}^{\infty} g'(a_n)(h) = g'(a)(h).$$

According to $g'(Y)$ is relatively compact, then from [24,25], we obtain $\lim_n g'(a_n) = g'(a)$.

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Proof: Assume that $(a_n)_n$ is a V -bdd sequence of weakly q -Cauchy type. Then $(g(a_n))_n$ is weakly Cauchy, $g(a_n)$ is a reasonably compact set. This suggests the norm convergence of $(g(a_n))_n$, which is easily demonstrated. Therefore $g \in C_{wsc}^q(V, B)$.