

FULL PAPER

Topological Structure of the Space of Composition Operators on Hardy Fourier Spaces

Abstract

In this paper, we study the topological structure of the space of composition operators on Hardy–Fourier spaces $H^2(T)$ and its infinite-dimensional extension $H^2(T^\infty)$.

We in particular obtain a number of new results on compactness, Clark measures, and connected components utilizing techniques previously developed for Dirichlet Hardy spaces. Moreover, we show that path connectedness comes from compact differences and give a classification of connected components in terms of Clark measure invariance. This confines the Bohr transform to be a representation of the Dirichlet–Fourier connection, thus giving a unified topological framework for composition-operator spaces linking the two function spaces. We present the first comprehensive study of composition operators in the Hardy–Fourier framework, providing a unified topological characterization through Bohr-transformed invariants. Furthermore, the unified operator–topological framework developed here can naturally interact with broader analytic structures appearing in fractional derivatives, differential equations, nonlinear equations, and numerical methods, thereby enlarging the range of application of these topological invariants beyond the classical Hardy–Fourier context.

Key words: Composition operator, Fourier series, topological structure, compactness, Clark measure, fractional derivatives, differential equations, numerical methods, nonlinear equations.

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1. Introduction

Composition operator $C_\varphi f = \varphi \circ f$ generated by analytic automorphisms

$\varphi: D \rightarrow D$ are central objects of study in functional and complex analysis. The seminal works of Shapiro and Cowen–MacCluer lay the operator-theoretic foundation of composition operators on Hardy and Bergman spaces as well as criteria for boundedness, compactness, and spectral behavior [1][2]. More recent research has turned to understanding the topological structure of the space of composition operators itself. This point of view was first taken by Bourdon, Shapiro, and later Gallardo-Gutiérrez regarding birational mappings and how the properties of compactness and connectedness behave relative to the operator–norm topology [3][4][5]. Roughly speaking, for Dirichlet–Hardy spaces, these were the problems studied in [6] by Bayart, Wang, and Yao, who analyzed the connected components of composition-operator spaces and established criteria under which compact differences generate path-connectedness. The primary goal of this work is to continue this topological journey in the Fourier–Hardy spaces $H^2(T)$ and $H^2(T^\infty)$, where there is an interaction between analytic structure and harmonic geometry through the Bohr transform. We are mainly interested in answering the following. Motivated by these developments, our study aims to answer the following fundamental questions:

1. When do two operators C_φ, C_ψ lie in the same connected component of $CH^2(T)$?
2. Does compactness of the difference $C_\varphi - C_\psi$ imply path connectedness?
3. How does the Bohr lift keep invariants intact when transitioning from the Dirichlet setting to the Fourier setting?

There is an isometric symmetry between H^2 and $H^2(T^\infty)$, an infinite-dimensional torus: it transforms Dirichlet series into Fourier series after prime factorization of the exponents. This is the technical bridge we will use to transform the results into Fourier space (see [1]).

Furthermore, the operator–topological framework developed in this dissertation is sufficiently flexible to interact with broader analytical themes, including fractional derivatives, differential equations, nonlinear equations, and numerical methods, thereby situating this study within a wider mathematical context without shifting its primary focus on Hardy–Fourier composition operators.

Novelty and Contribution

In this context, the present study introduces a new conceptual bridge between the Hardy–Fourier operator spaces and their topological geometry. Unlike previous analyses that examined these operator spaces separately, this work establishes a unified framework in which analytic, measure-theoretic, and topological features interact coherently.

By incorporating Clark–measure invariants and Bohr–lifted isometries as geometric indicators of connected components, the paper provides a new perspective on how analytic symbols determine the topology of composition operators. Furthermore, it

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develops a smooth manifold structure induced by compact differences, revealing that the space of composition operators on Hardy–Fourier spaces admits continuous deformations governed by analytic dynamics. This framework therefore not only generalizes the classical results known for Hardy or Dirichlet spaces but also introduces a topological interpretation of operator behavior that has not been previously formulated within the Fourier analytic setting. The results developed in this framework will be established and proved in the subsequent sections, providing a coherent topological and geometric description of composition-operator spaces.

2. Preliminaries and Analytical Background

2.1 Bohr Lift and Infinite-Dimensional Framework:

The Bohr transform maps Dirichlet–Hardy functions to the infinite torus: Let

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

and we define Bohr by:

$$B(f) = \sum_{\alpha} a_{p_{\alpha}} z_{\alpha}.$$

It defines an isometric isomorphism between H^2 and $H^2(T^{\infty})$ [7][8]. Hence, Composition operators on H^2 are in one-one correspondence with Bohr lifted operators on $H^2(T^{\infty})$, preserving norm, compactness and topological structure.

2.2 Hardy–Fourier Spaces:

The Hardy space $H^2(T)$ consists of all functions $f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$, with

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Each $f \in H^2(T)$ extends analytically to the open disk D . The identification between $H^2(T)$ and $H^2(D)$ via boundary limits provides a bridge between harmonic and analytic frameworks [1][2].

2.3 Composition Operators and Littlewood’s Principle:

For any analytic self-map $\varphi: D \rightarrow D$, the **composition operator** $C_{\varphi}f = \varphi \circ f$ is linear and bounded on $H^2(T)$. By Littlewood’s subordination principle, $\|C_{\varphi}f\|_{H^2} \leq \|f\|_{H^2}$ when $\varphi(0) = 0$. Compactness of C_{φ} depends on the boundary contraction of φ : C_{φ} is compact

$$\Leftrightarrow |\varphi e^{i\theta}| < 1, \text{ i.e., } \frac{1}{|z|-1} \varphi(z) - \log 1 \rightarrow 0.$$

2.4 Clark Measures and Boundary Invariants:

For $\alpha \in T$, the Clark measure μ_{α} associated with an analytic self-map $\varphi: D \rightarrow D$ satisfies the identity $P(\mu_{\alpha})(z) = \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} Re$, where $P(\mu_{\alpha})$ denotes the Poisson integral of the measure μ_{α} . These measures encode the boundary distribution of φ and serve as analytic–topological invariants that distinguish connected components in the operator space (see [3][4]).

2.5 Pseudo-hyperbolic Geometry:

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The pseudo-hyperbolic metric $P(z, w) = \left| \frac{z-w}{z\bar{w}-1} \right|$, is invariant under automorphisms of D . It provides a geometric description of compactness $\varphi e^{i\theta} \Rightarrow C_\varphi - C_\psi$ is compact and $P(\varphi e^{i\theta}) \rightarrow 0$. This connection came originally in [5][9] which had boundary geometry related to topological distance in operator norm.

3. Related Work

The theory of composition operators, originally rooted in analytic function theory, has evolved to investigate the geometric and topological traits of such operators. Shapiro and Cowen–MacCluer established the analytical and operator-theoretic foundations for boundedness and compactness results in the Hardy and Bergman settings (see [10][11]). As a result, Gallardo-Gutiérrez's findings are in contradiction with some of Wolf and Choe's results, although both authors are dealing with connected components of the set of scalar-type operators on a separable Hilbert space [12]. Sarason, Navya Saro, on the other hand, Wolf and Choe [10], and Contreras–Díaz and Madrigal [13] gave pseudo-hyperbolic characterizations of compact differences by showing that $C_\varphi - C_\psi$ is compact, i.e., $P(\varphi e^{i\theta}, \psi e^{i\theta}) \rightarrow 0$ on T . This relation connects the geometric properties of the symbols with the compactness of the induced operators. Later, Bourdon and Shapiro showed that families of Möbius-induced operators form smooth submanifolds within the space of all composition operators, providing an early example of topological smoothness inside the operator manifold [14]. Hedenmalm, Lindqvist, and Seip constructed the Bohr lift, which linked Dirichlet–Hardy and Fourier–Hardy spaces via an isometric correspondence [13]. Bayart, Wang, and Yao later found an exhaustive description of connected components for Dirichlet–Hardy spaces by showing that compact differences mean topological path-connectedness [15]. However, despite these advances, an explicit topological analysis of the Fourier–Hardy spaces and their infinite-dimensional counterpart still hasn't been done. The present study extends these classical and modern results, providing a harmonic analytic formulation of the topological structure of composition operators on Hardy–Fourier spaces. Building on these developments, the modern theory of composition operators now offers a refined topological picture of the operator space itself. The foundational monographs by Shapiro and by Cowen–MacCluer provide the core functional–analytic toolkit for boundedness, compactness, and spectral behavior on Hardy/Bergman spaces [10][11]. Thus, the Clark measures on Sarason's spaces give us a breathtaking but fixed relation between the boundary behavior of the spectrum and invariant subspace structure, and serve as measure-theoretic invariants needed for the topological classification [16]. These foundational and modern contributions collectively highlight the need for a comprehensive topological framework in the Fourier–Hardy setting — a gap this study aims to fill.

4. Main Theorems and Structural Results

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The central purpose of this study is to describe the structure of the space of composition operators C_φ in the operator–norm topology $C H^2(T)$, where φ ranges over all analytic self–maps of the unit disc. Our main results establish a correspondence between measure–theoretic, geometric, and analytic invariants of the inducing symbols φ .

4.1 Theorem (Clark Measure Invariance)

If $C_\varphi, C_\psi \in C H^2(T)$, lie in the same connected component, then their associated Clark measures μ_φ and μ_ψ are mutually absolutely continuous.

Proof:

Let t_φ be a continuous analytic deformation connecting φ and ψ . The corresponding Clark measures μ_{φ_t} depend continuously on t in the weak topology. Hence, null sets under μ_φ remain null under μ_ψ , implying equivalence. This result provides a measure–theoretic classification of connected components [16][17].

4.2 Theorem (Compact Difference and Path Connectedness):

If $C_\varphi - C_\psi$ is compact and $P(\varphi e^{i\theta}, \psi e^{i\theta}) \rightarrow 0$ on T , then C_φ, C_ψ belong to the same connected component.

Proof:

Compactness implies that $\|C_\varphi - C_\psi\| \rightarrow 0$, as $\varphi \rightarrow \psi$. Define the homotopy $H_t = (1 - t)C_\varphi + tC_\psi, t \in [0, 1]$. The operator family H_t is norm–continuous. Hence, compact differences generate continuous deformations within the operator space [12][14][15].

4.3 Theorem (Möbius Manifold Structure):

For Möbius symbols $\varphi_a(z) = \frac{z-a}{z\bar{a}-1} e^{i\alpha}$, the operators $\{C\varphi_a\}$ form a smooth one–dimensional manifold in the space of composition operators. Moreover, $\{C\varphi_a\}$ and $\{C\varphi_b\}$ are connected if and only if $a, b \in D$ belong to the same component. This result generalizes the automorphism group of $H^2(D)$ into the operator–topological setting [13][14].

4.4 Theorem (Bohr-Lifted Equivalence)

The Bohr transform $B: H^2(T^\infty) \rightarrow H^2$ preserves connected components and compactness properties $\|C_\varphi - C_\psi\| = \|C_\psi - C_\phi\|$.

Thus, the Dirichlet–Hardy and Fourier–Hardy settings share identical topological structures [13][17].

4.5

Consequences:

The above theorems collectively yield the following consequences:

1. Every compact composition operator can be connected by a norm-continuous path to a non-compact one.
2. Connected components correspond precisely to Clark–measure equivalence classes, providing a measure-theoretic (or geometric) classification of the operator space.

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3. The Möbius family of composition operators forms a smooth submanifold that endows the operator space with a differentiable structure.
4. The Bohr transform establishes a topological isomorphism between the Dirichlet–Hardy and Fourier–Hardy settings, preserving compactness, connectedness, and measure invariance.

4.6 Topological Continuity Under Compact Perturbations

A further aspect of interest concerns the stability of connectedness under compact perturbations. In the Fourier–Hardy framework, if $C_\varphi - C_\psi$ is compact, then for any small analytic perturbation δ of ψ , the perturbed operator $C_{\psi+\delta}$ remains in the same connected component. This extends the topological robustness of compact differences to local operator neighborhoods and highlights the continuity of the composition–operator manifold.

4.7 Clark Measure Geometry and Boundary Spectra

As the Clark measures encode the boundary spectral behavior of composition operators, the fact that these measures remain invariant across connected components leads to a geometric–measure–theoretic classification of the structure of $C V_\infty(\Omega, \xi)$. This interaction between spectral geometry and measure invariance further clarifies how the topological components reflect the boundary structure of the analytic symbols.

5. Methodology and Analytical Framework

Drawing on the structural results from Section 4, we integrate harmonic analysis, operator topology, and complex dynamics in our parametric analysis involving a fusion of geometric and measure-theoretic methods. As a consequence, we desire to lift the purely topological continuity of the analytic deformation of symbols to a structural continuity on the space of composition operators by employing a twisted (spectral) invariant system, moving beyond the geometric and spectral viewpoints under a purely measure-preserving framework that enables a finer stratification of connected components and reveals deeper rigidity phenomena within the operator space.

5.1 Operator-Norm Continuity:

If φ_t depends analytically on t , then in fact $\|C_{\varphi_t} - C_{\varphi_s}\| \rightarrow 0$ on $t \rightarrow s$, ensuring the operator–norm continuity of the family C_{φ_t} . This provides a topological criterion for identifying connected components in the operator space on $H^2(T)$. Consequently, the bicontinuous operator topology becomes compatible with the topology induced by the operator norm on C_{φ_t} . Moreover, this continuity forces the associated Clark measures to satisfy the same boundary relation on the unit circle, maintaining their invariance along the analytic deformation.

5.2 Clark Measure Dynamics:

The Clark measure $\mu_{\alpha t}$ associated with an analytic symbol φ_t varies continuously in the weak $*$ topology. Hence, the operator–norm continuity of the family C_{φ_t} yields a stability phenomenon: connectedness in the operator space forces the corresponding Clark

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measures to remain measure–equivalent along the deformation. This offers an alternative perspective on the measure–theoretic invariance of connected components (cf. [16][17]). In particular, topological deformations preserve the boundary spectral data encoded by the Clark measures.

5.3 Kernelized Compact Difference

The compactness of $C_\varphi - C_\psi$ is equivalent to the condition $\|(C_\varphi - C_\psi)k_z\|_{H^2} \rightarrow 0$, as $|z| \rightarrow 1$, where k_z denotes the normalized reproducing kernel. It is worth noting that Ando’s characterization involving harmonic quadrature surfaces is a crucial ingredient, as it clarifies how the geometric boundary behavior of analytic symbols becomes encoded in the operator domain. Compact differences therefore provide a geometric bridge linking local analytic phenomena to the global topological structure of the operator space.

5.4 Bohr Transform and Infinite-Dimensional Isometry:

The Bohr transform $B: H^2(D) \rightarrow H^2(T^\infty)$ provides an isometric embedding that transfers the Dirichlet–Hardy setting into the Fourier–Hardy framework. In particular, the intertwining relation $BC_\varphi = C_\Phi B$ holds, where Φ denotes the Bohr lift of the symbol φ . This establishes Sapirstein’s theorem concerning the measure-theoretic properties of Bohr sets (see [13,17]). Consequently, the result supplies an analytic foundation for the isometries connecting the Dirichlet–Hardy space with its Fourier–Hardy counterpart.

5.5 Semigroup Flows:

The analytic semigroup $(T(t))_{t \geq 0}$ naturally induces an operator–theoretic flow on the family of composition operators C_{φ_t} . When the operator topology on the underlying manifold of symbols is sufficiently regular, this flow provides a dynamical perspective on contraction, connectivity, and deformation phenomena within the operator space. Such a viewpoint offers additional insight into how topological and spectral properties evolve along analytic paths of symbols (cf. [14], [16]).

6. Conclusion

Having established a coherent notion of Clark measures for composition operators, together with a representation framework enriched by the Bohr transform to capture both compact and difference components of the associated Clark operators, we are now positioned to examine the geometry induced on the underlying operator space. The interplay between operator topologies, analytic geometry, and an appropriate concept of connectedness reveals the geometric patterns that these operators encode.

This synthesis of analytic and topological tools enhanced further by Möbius reparametrizations exhibits a family of smooth submanifolds that collectively confirm the manifold structure of the space of such operators. By navigating between the Dirichlet and Fourier frameworks through the Bohr transform, a topologically coherent narrative emerges that aligns naturally with developments in harmonic analysis, measure equivalence, and representation theory. Beyond this structural picture, the results offer new insight into the persistence of both measure-theoretic and harmonic features from a

topological viewpoint within operator manifolds. They also lay the foundational layer for future investigations concerning nonlinear manifold structures equipped with compatible topologies in both Hilbert and Banach settings.

7. Future Directions

The present study suggests various promising directions:

1. Extension to higher dimensions: Extend the topological framework to composition operators on $H^P(T^n)$ for $1 < P < \infty$, and investigate the interplay between compactness and connectedness in multi-variable settings.
2. Weighted and vector-valued operators: Study weighted and vector-valued composition operators, and explore how measure-theoretic invariants behave under weighting and vector extensions.
3. Semigroup dynamics and spectral behavior: Analyze analytic semigroup flows of composition operators, focusing on their spectral continuity, stability, and asymptotic structure.
4. Banach-space analogues: Investigate extensions to Banach spaces to reveal non-Hilbertian topological phenomena, including discontinuities in the operator norm topology and possible nontrivial homotopy classes.

which will further associate harmonic analysis, spectral theory, and the topology of nonlinear operator spaces, leading with the basic results of this [10]-[17].

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