The Scholar Journal for Sciences & Technology sjst

ISSN (Print) 2794-7629
ISSN (Online) 2794- 4549

Received 27/ 05/2024
Accepted 17/06/2024

FULL PAPER

Two Results to Clarify the Relationship Between P^h_{ijk} and R^h_{ijk} with Two Connections of Third Order in Finsler Spaces

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Abstract

The relations between various curvature tensors discussed by Finslerian geometrics. In this paper, two theorems that clarify the relationship between P^h_{ijk} and R^h_{ijk} in generalized tri-recurrent space are discussed. Moreover, the behavior of some tensors are tri-recurrent if R^h_{ijk} satisfies the generalized tri-recurrence property.

Keywords: Holomorphic ally projective curvature tensorP^h_{ijk} , $\mathcal{B}$ $-$covariant derivative, h $-$and covariant derivative.

1. Introduction and Preliminaries

Finsler geometry is a kind of differential geometry. It usually consider as a generalization of Riemannian geometry. In this research, we use a methodology based on previous theories and spaces in Finsler geometry. Here we focus on the relationship between the Holomorphically projective curvature tensor P^h_{ijk} and curvature tensor P^h_{ijk}.

The importance of the research lies in studying this relationship and developing it in more general spaces with two connections.

The relationship between the curvature tensors R^h_{ijk}, W^h_{ijk} and P^h_{ijk} in generalized recurrent and birecurrent Finsler space in sense of Berwald discussed by [1, 5]. Srivastava [17] defined special R-generalized recurrent Finsler spaces of the order two.

Further, the generalized $\mathcal{B}P$ −tri-recurrent space and generalized P^h −tri-recurrent space introduced by [2, 3]. The generalized P^h −tri-recurrent space discussed by Qasem et al. [13].

An n −dimensional Finsler space F_n equipped with the metric function $F(x,y)$ satisfying the three conditions [6, 18]. The vector y_i , g_{ij} and g^{ij} are defined by

(1.1) $y_i = g_{ij}(x,y)\, y^j$.

(1.2) $g_{ij}\, g^{ik} = \delta_j^k = \begin{cases} 1 & if\;\; j = k\,, \\ 0 & if\;\; j \neq k. \end{cases}$

In view of Eqs. (1.1) and (1.2), we have

(1.3) $\delta_k^i y^k = y^i$, $\delta_j^i g_{ir} = g_{jr}$ and $\delta_k^i y_i = y_k$.

The tensor C_{ijk} is defined by [8]

$$C_{ijk} = \frac{1}{2}\, \dot{\partial}_i\, g_{jk} = \frac{1}{4}\, \dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2 \quad .$$

The above tensor satisfies the following [8, 10, 16]

(1.4) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ and $\delta_k^i C_{jil} = C_{jkl}$.

Berwald introduced a covariant derivative, this derivative denoted by $\mathcal{B}_k$. This derivative with tensor T_j^i is defined by

$$\mathcal{B}_k T_j^i = \partial_k\, T_j^i - \left(\dot{\partial}_r T_j^i\right) G_k^r + T_j^r\, G_{rk}^i - T_r^i\, G_{jk}^r \,.$$

The covariant differentiation in Berwald sense of the vector y^i and metric tensor g_{ij} satisfy

(1.5) $\mathcal{B}_k\, y^i = 0$ and $\mathcal{B}_k g_{ij} = -2\, y^h\, \mathcal{B}_h\, C_{ijk} = -2\, C_{ijk|h}\, y^h$.

Cartan h −covariant derivative with x^k is given by [7, 15]

$$X^i_{(k)} = \dot{\partial}_k\, X^i - \left(\, \dot{\partial}_r\, X^i\right) G_k^r + X^r\, \Gamma_{rk}^{*i} \;\; ,$$

where Γ_{rk}^{*i} is a function called it Cartan's connection parameter. Therefore, the covariant differentiation in Cartan sense of y^i and g_{ij} are vanish identically i.e.

(1.6) $y^i_{(k)} = 0$ and $g_{ij(k)} = 0$.

The curvature tensor R_{jkh}^i is defined by [15]

$$R_{jkh}^i = \partial_h \Gamma_{jk}^{*i} + \left(\, \dot{\partial}_l \Gamma_{jk}^{*i}\,\right) G_k^l + \; C_{jm}^i \left(\, \partial_h G_k^m - G_{hl}^m\, G_k^l\right) + \Gamma_{mk}^{*i}\, \Gamma_{jh}^{*m} - h/k\,,$$

which satisfies

(1.7) $R_{jki}^i = R_{jk}$.

The tensor R_{jkh}^i , its associative R_{rjkh} , R–Ricci tensor R_{jk} and curvature vector R_k satisfy [11, 12]

(1.8) $R_{rjkh} = g_{ri}\, R_{jkh}^i$, $R_{jk} y^j = R_k$ and $R_{jkh}^i y^j = H_{kh}^i = K_{jkh}^i y^j$.

The tensor P_{jkh}^i called hv −curvature tensor is defined by [4, 9, 15]

$$P_{jkh}^i = \dot{\partial}_h \Gamma_{jk}^{*i} + C_{jr}^i P_{kh}^r - C_{jh|k}^i \,.$$

The associate tensor P_{ijkh}, torsion tensor P^i_{kh} and P –Ricci tensor P_{jk} of hv –curvature tensor P^i_{jkh} satisfies the relations

(1.9) $\quad P_{ijkh} = g_{ir}\, P^r_{jkh}, \quad P^i_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r, \quad P^i_{jki} = P_{jk} \quad$ and $\quad P^i_{jk} y_i = 0.$

The generalized tri-recurrent space in Berwald and Cartan senses which are characterized by the conditions [2, 3]

(1.10) $$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P^i_{jkh} = a_{lmn} P^i_{jkh} + b_{lmn}\left(\delta^i_j g_{kh} - \delta^i_k\, g_{jh}\right) - 2y^t c_{mn}\mathcal{B}_t\left(\delta^i_j C_{khl} - \delta^i_k\, C_{jhl}\right) - 2y^t d_{ln}\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k\, C_{jhm}\right) - 2y^t \mu_n \mathcal{B}_l\mathcal{B}_t\left(\delta^i_j C_{khm} - \delta^i_k\, C_{jhm}\right),$$

(1.11) $$P^i_{jkh(l)(m)(n)} = c_{lmn} P^i_{jkh} + d_{lmn}\left(\delta^i_j g_{kh} - \delta^i_k\, g_{jh}\right),$$

respectively, where $a_{lmn} = \mathcal{B}_l u_{mn} + u_{mn}\lambda_l$ and $b_{lmn} = \mathcal{B}_l v_{mn} + u_{mn}\mu_l$ are non – zero covariant tensors field of third order, $c_{mn} = v_{mn}$ and $d_{ln} = \mathcal{B}_l \mu_n$ are non – zero covariant tensor field of second order. $\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n$ is the differential operator in Berwald sense with x^n, x^m and x^l, successively. Also, $c_{lmn} = a_{lm(n)} + a_{lm}\lambda_n$ and $d_{lmn} = a_{lm}\mu_n + b_{lm(n)}$ are non – zero covariant tensors field of third order. $(l)(m)(n)$ is the differential operator in sense of Cartan with x^l, x^m and x^n, successively. The mentioned spaces are denoted them by $G(\mathcal{B}P) - TRF_n$ and $G(P^h) - TRF_n$. Also, the tensors which satisfy the conditions (1.10) and (1.11) are called generalized $\mathcal{B}$ –tri-recurrent tensor and generalized h –tri-recurrent tensor.

Transvecting the condition (1.10) by y^j, applying Eqs. (1.3), (1.4), (1.5) and (1.9), we get

(1.12) $$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n P^i_{kh} = a_{lmn} P^i_{kh} + b_{lmn}\left(y^i g_{kh} - \delta^i_k\, y_h\right) - 2y^t c_{mn}\mathcal{B}_t\left(y^i C_{khl}\right) - 2y^t d_{ln}\mathcal{B}_t\left(y^i C_{khm}\right) - 2y^t \mu_n \mathcal{B}_l\mathcal{B}_t\left(y^i C_{khm}\right).$$

Contracting i and h in the condition (1.10), applying Eqs. (1.3), (1.4) and (1.9), we get

(1.13) $$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\, P_{jk} = a_{lmn}\, P_{jk}\,.$$

Transvecting (1.11) by g_{ir}, using Eqs. (1.3), (1.6) and (1.9), we get

(1.14) $$P_{rjkh(l)(m)(n)} = c_{lmn}\, P_{rjkh} + d_{lmn}\left(g_{jr} g_{kh} - g_{kr} g_{jh}\right).$$

Transvecting (1.11) by y^j, using Eqs. (1.1), (1.3), (1.6) and (1.9), we have

(1.15) $$P^i_{kh(l)(m)(n)} = c_{lmn} P^i_{kh} + d_{lmn}\left(y^i g_{kh} - \delta^i_k\, y_h\right).$$

Contracting i and h in the condition (1.10), using Eqs. (1.3) and (1.9), we have

(1.16) $$P_{jk(l)(m)(n)} = c_{lmn} P_{jk}\,.$$

2. Main Results

In this section, we focus on the conditions for R^i_{jkh} that be generalized tri-recurrent in two senses. The holomorphically projective curvature tensor P^h_{ijk} is defined by [14]

(2.1) $$P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2S_{ij}F^h_k\right),$$

where $S_{ij} = F^a_i\, R_{aj}$. Transvecting Eq. (2.1) by g_{hr}, using Eqs. (1.3), (1.8), (1.9) and put $(g_{hl}F^h_j = g_{jl})$, we get

(2.2) $$P_{rijk} = R_{rijk} + \frac{1}{n+2}\left(R_{ik}\, g_{jr} - R_{jk}\, g_{ir} + S_{ik}\, g_{jr} - S_{jk}\, g_{ir} + 2\, S_{ij}\, g_{kr}\right).$$

Transvecting Eq. (2.1) by y^i, using Eqs. (1.3), (1.8) and (1.9), we get

(2.3) $$P^h_{jk} = H^h_{jk} + \frac{1}{n+2}\left(R_k\delta^h_j - R_{jk}\, y^h + S_kF^h_j - S_{jk}\, y^h + 2\, S_jF^h_k\right).$$

Contracting i and h in Eq. (2.1), using Eqs. (1.2), (1.7), (1.9) and put $(S_{ik}F^i_j = S_{jk})$, we get

(2.4) $$P_{jk} = R_{jk} + \frac{1}{n+2}\left[(1-n)R_{jk} + (3-n)\, S_{jk}\right].$$

Theorem 2.1. *In $G(\mathcal{B}P) - TRF_n$, the tensor $\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2\, S_{ij}F^h_k\right)$ behaves as tri-recurrent if R^h_{ijk} is a generalized $\mathcal{B}$ −tri-recurrent tensor.*

Proof. Assume that a $G(\mathcal{B}P) - TRF_n$, i.e. characterized by the condition (1.10). Taking $\mathcal{B}$ −covariant derivative for Eq. (2.1) thrice with respect to x^n, x^m and x^l, then applying the condition (1.10), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_nR^h_{ijk} = a_{lmn}P^h_{ijk} + b_{lmn}\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right) - 2y^t c_{mn}\mathcal{B}_t\left(\delta^h_i C_{jkl} - \delta^h_j C_{ikl}\right)$$
$$-2y^t d_{ln}\mathcal{B}_t\left(\delta^h_i C_{jkm} - \delta^h_j C_{ikm}\right) - 2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^h_i C_{jkm} - \delta^h_j C_{ikm}\right)$$
$$-\frac{1}{n+2}\,\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2\, S_{ij}F^h_k\right).$$

Using Eq. (2.1) in above equation, we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\, R^h_{ijk} = a_{lmn}R^h_{ijk} + b_{lmn}\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right) - 2y^t c_{mn}\mathcal{B}_t\left(\delta^h_i C_{jkl} - \delta^h_j C_{ikl}\right)$$
$$-2y^t d_{ln}\mathcal{B}_t\left(\delta^h_i C_{jkm} - \delta^h_j C_{ikm}\right) - 2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(\delta^h_i C_{jkm} - \delta^h_j C_{ikm}\right)$$

if and only if

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2\, S_{ij}F^h_k\right)$$
$$= a_{lmn}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2\, S_{ij}F^h_k\right).$$

The above equation refers to required.

Now, we have two corollaries related to the previous theorem. Taking covariant differentiation in Berwald sense for Eq. (2.3) thrice with x^n, x^m and x^l, then applying Eq. (1.12), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_nH^h_{jk} = a_{lmn}P^h_{jk} + b_{lmn}\left(y^h g_{jk} - \delta^h_j y_k\right) - 2y^t c_{mn}\mathcal{B}_t\left(y^h C_{jkl}\right)$$

$$-2y^t d_{ln}\mathcal{B}_t\left(y^h C_{jkm}\right) - 2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(y^h C_{jkm}\right)$$

$$-\frac{1}{n+2}\,\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(R_k\delta_j^h - R_{jk}\,y^h + S_kF_j^h - S_{jk}\,y^h + 2\,S_jF_k^h).$$

Using Eq. (1.10) in above equation, we get

(2.5) $$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_nH_{jk}^h = a_{lmn}H_{jk}^h + b_{lmn}\left(y^h g_{jk} - \delta_j^h y_k\right) - 2y^t c_{mn}\mathcal{B}_t\left(y^h C_{jkl}\right)$$

$$-2y^t d_{ln}\mathcal{B}_t\left(y^h C_{jkm}\right) - 2y^t\mu_n\mathcal{B}_l\mathcal{B}_t\left(y^h C_{jkm}\right)$$

if and only if

(2.6) $$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n(R_k\delta_j^h - R_{jk}\,y^h + S_kF_j^h - S_{jk}\,y^h + 2\,S_jF_k^h)$$

$$= a_{lmn}(\,R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}\delta_i^h + 2\,S_{ij}F_k^h).$$

The equation (2.6) refers to required. Thus, we conclude

Corollary 2.1. *The tensor* $(\,R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}\delta_i^h + 2\,S_{ij}F_k^h)$ *behaves as tri-recurrent if the covariant differentiation in Berwald sense of third order for* H_{jk}^h *is given by Eq. (2.5) in* $G(\mathcal{B}P) - TRF_n$.

Taking $\mathcal{B}$ −covariant derivative for Eq. (2.4) thrice with respect to x^n, x^m and x^l, then applying Eq. (1.13), we get

$$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\,R_{jk} = a_{lmn}P_{jk} - \frac{1}{n+2}\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\,[(1-n)R_{jk} + (3-n)\,S_{jk}]\,.$$

Using Eq. (2.4) in above equation, we get

(2.7) $$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_nR_{jk} = a_{lmn}R_{jk}$$

if and only if

(2.8) $$\mathcal{B}_l\mathcal{B}_m\mathcal{B}_n\,[(1-n)R_{jk} + (3-n)S_{jk}] = a_{lmn}\,[(1-n)R_{jk} + (3-n)S_{jk}].$$

From equations (2.7) and (2.8), we conclude

Corollary 2.2. *The behavior* R *−Ricci tensor* R_{ik} *is tri-recurrent if and only if the tensor* $[(1-n)R_{jk} + (3-n)\,S_{jk}]$ *behaves as tri-recurrent in* $G(\mathcal{B}P) - TRF_n$.

In the following theorem, we infer the condition for R_{ijk}^h that be generalized tri-recurrent in Cartan sense.

Theorem 2.2. *In* $G(P^h) - TRF_n$ *, the tensor* $(\,R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}\delta_i^h + 2\,S_{ij}F_k^h)$ *behaves as tri-recurrent if* R_{ijk}^h *is a generalized* h *−tri-recurrent tensor.*

Proof. Let us consider a $G(P^h) - TRF_n$, i.e, characterized by the condition (1.11). Taking covariant differentiation in Cartan sense for Eq. (2.1) thrice with x^n, x^m and x^l, then applying the condition (1.11), we get

$$R_{ijk(l)(m)(n)}^h = c_{lmn}P_{ijk}^h + d_{lmn}\left(\delta_i^h g_{jk} - \delta_j^h g_{ik}\right)$$

$$-\frac{1}{n+2}(R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}\delta_i^h + 2\,S_{ij}F_k^h)_{(l)(m)(n)}.$$

Using Eq. (2.1) in above equation, we get

$$R^h_{ijk(l)(m)(n)} = c_{lmn}P^h_{ijk} + d_{lmn}\left(\delta^h_i g_{jk} - \delta^h_j g_{ik}\right)$$

if and only if

$$(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2\, S_{ij}F^h_k)_{(l)(m)(n)}$$
$$= c_{lmn}\,(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}\delta^h_i + 2\, S_{ij}F^h_k).$$

The above equation refers to required.

Now, we have two corollaries related to the previous theorem. Taking covariant differentiation in Cartan sense for Eq. (2.2) thrice with x^n, x^m and x^l, then applying Eq. (1.14), we have

$$R_{rjkh(l)(m)(n)} = c_{lmn}\,P_{rjkh} + \mu_m\left(g_{ir}g_{jk} - g_{jr}g_{ik}\right)$$
$$-\frac{1}{n+2}(R_{ik}\, g_{jr} - R_{jk}\, g_{ir} + S_{ik}\, g_{jr} - S_{jk}\, g_{ir} + 2\, S_{ij}\, g_{kr})_{(l)(m)(n)}.$$

Using Eq. (2.2) in above equation, we infer

(2.9) $$R_{rjkh(l)(m)(n)} = c_{lmn}R_{rjkh} + d_{lmn}\left(g_{ir}g_{jk} - g_{jr}g_{ik}\right)$$

if and only if

(2.10) $$(R_{ik}\, g_{jr} - R_{jk}\, g_{ir} + S_{ik}\, g_{jr} - S_{jk}\, g_{ir} + 2\, S_{ij}\, g_{kr})_{(l)(m)(n)}$$
$$= c_{lmn}\left(R_{ik}\, g_{jr} - R_{jk}\, g_{ir} + S_{ik}\, g_{jr} - S_{jk}\, g_{ir} + 2\, S_{ij}\, g_{kr}\right).$$

Taking covariant differentiation in Cartan sense for Eq. (2.3) thrice with respect to x^n, x^m and x^l, then applying Eq. (1.15), we get

$$H^h_{jk(l)(m)(n)} = c_{lmn}P^h_{jk} + d_{lmn}\left(y^h g_{jk} - \delta^h_j y_k\right)$$
$$-\frac{1}{n+2}(R_k\delta^h_j - R_{jk}\, y^h + S_kF^h_j - S_{jk}\, y^h + 2\, S_jF^h_k)_{(l)(m)(n)}.$$

Using Eq. (2.3) in previous equation, we get

(2.11) $$H^h_{jk(l)(m)(n)} = c_{lmn}\, H^h_{jk} + d_{lmn}\left(y^h g_{jk} - \delta^h_j y_k\right)$$

if and only if

(2.12) $$(R_k\delta^h_j - R_{jk}\, y^h + S_kF^h_j - S_{jk}\, y^h + 2\, S_jF^h_k)_{(l)(m)(n)}$$
$$= c_{lmn}\left(R_k\delta^h_j - R_{jk}\, y^h + S_kF^h_j - S_{jk}\, y^h + 2\, S_jF^h_k\right).$$

From equations (2.10) and (2.12), we conclude

Corollary 2.3. *In* $G(P^h) - TRF_n$, *the tensors* $\left(R_{ik}\, g_{jr} - R_{jk}\, g_{ir} + S_{ik}\, g_{jr} - S_{jk}\, g_{ir} + 2\, S_{ij}\, g_{kr}\right)$ *and* $\left(R_k\delta^h_j - R_{jk}\, y^h + S_kF^h_j - S_{jk}\, y^h + 2\, S_jF^h_k\right)$ *behave as tri-recurrent if the covariant differentiation in Cartan sense of third order for* R_{rjkh} *and* H^h_{jk} *are satisfied by Eqs.(2.9) and (2.11), respectively.*

Taking covariant differentiation in Cartan sense for Eq. (2.4) thrice with respect to x^n, x^m and x^l, then applying Eq. (1.16), we infer

$$R_{jk(l)(m)(n)} = c_{lmn} P_{jk} - \frac{1}{n+2} [(1-n)R_{jk} + (3-n)\, S_{jk}]_{(l)(m)(n)} \; .$$

Using Eq. (2.4) in previous equation, we get

(2.13) $\quad R_{jk(l)(m)(n)} = c_{lmn} R_{jk}$

if and only if

(2.14) $\quad [(1-n)R_{jk} + (3-n)\, S_{jk}]_{(l)(m)(n)} = c_{lmn} [(1-n)R_{jk} + (3-n)\, S_{jk}]$.

From equations (2.14), we conclude

Corollary 2.4: *The behavior* R *−Ricci tensor* R_{ik} *is tri-recurrent if the tensor* $[(1-n)R_{jk} + (3-n)\, S_{jk}]$ *behaves as tri-recurrent in* $G(P^h) - TRF_n$.

3. Conclusions

The condition for R^h_{ijk} which be generalized tri-recurrent tensor in $G(\mathcal{B}P) - TRF_n$ and $G(P^h) - TRF_n$ has been obtained. Further, various identities related to above mentioned spaces have been studied.

4. Significance of Study

In previous study, we discussed the relationship between P^h_{ijk} and R^h_{ijk} in generalized recurrent and bi-recurrent Finsler spaces in Berwald sense. The importance of this study is to find a generalization and study this relationship in a generalized recurrent Finsler space of third order with two connections.

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