

Studying the Relationship.....

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FULL PAPER

Studying the Relationship and Computation of Cohomologies in Homogeneous Spaces.

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Abstract:

The aim of this paper is to analyze the relationship between fiber bundle theory and characteristic classes in homogeneous spaces. It clarifies the mathematical framework and definitions, utilising tools like classification spaces, homology, and cohomology theory to explain key concepts in algebraic topology. It highlights vector bundle theory as essential for these applications and describes the classification of fiber vector bundles, linking it to the characteristic class theory of cohomology for computations across various homogeneous spaces. Important definitions and properties of general vector bundles, as well as tangential and perpendicular bundles of smooth manifolds, are also discussed.

KEYWORDS: Fiber, vector bundles, characteristic classes, cohomology, homology theory, topological, homogeneous space, Sequence Analysis,

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Introduction:

Algebraic topology is a rich and dynamic field that connects geometry and algebra. To advance the traditional understanding of characteristic classes and about how to relate and calculate the cohomologies of different homogeneous spaces, vector bundle theory represents the common language between abstract algebraic constants and real-world geometric applications. Our method uses to offer standard axiomatically defined characteristic classes, such as the Steffell-Whitney, Chern, Pontryagin, and Euler classes,[14] provide fundamental information about bundles, such as their directionality, complex structure, and symmetry properties, which have wide-ranging applications in both pure mathematics and theoretical physics, particularly in gauge theory. By studying how these algebraic constants relate to one another, we can gain deeper insights into the geometric and topological aspects of space.[12] While simultaneously computing the cohomology(H) of the necessary classifying spaces. By navigating classifying spaces, we must compute the cohomology of different homogeneous spaces and classical groups to allow us to add applications of topological algebra. By looking at how these algebraic invariants are related to each other, the researchers can learn more about the geometric and topological aspects of space, which will help both mathematics and physics make more progress. [12] Furthermore, the calculation and classification of this symmetry constants are in an inseparable way linked to fiber bundle theory and space classification. These algorithms provide us not only with vector bundle sorting but also with a wealth of insights into various topological phenomena, focus on studying characteristic classes, we discover connections between them and fields such as differential and algebraic geometry, gouge theory, and even robotics. This demonstrates the importance and the feedback of these mathematical concepts, the structure of spectral sequences, particularly shothiserre spectral sequences, enables the systematic computation of symmetry smooth loops related to complex fiber bundles. These loops appear polynomial algebraic structures that define and constrain the behavior of characteristic classes.

1- Bundle theory and Characteristic classes.

In mathematics there are structures known as fiber bundles, a fiber is locally attached to a base space to form total space, and characteristic classes are an element in algebraic topology used to detect twist or non-triviality in a fiber bundle associated to cohomology class. [8]

In this context, we outline the key definitions that facilitate connections within the text.

Fiber bundle:

A fiber bundle consists of three spaces (E, B, F) with projection map $p: E \rightarrow B$ where E is the whole space, B defines the foundational space

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and F is the fiber which is attached to each point of base B .

Serre Spectral Sequence.

The Serre Spectral Sequence is a computational tool associated with a fiber bundle: $F \xrightarrow{i} E \xrightarrow{p} B$. [16]

It provides a method to reconstruct the cohomology elements of E from the cohomology elements of B and F .

Consider a cylinder as a fiber bundle. The foundation space B is a circle S^1 , Furthermore, the fiber F forms a line segment. The total space E is the cylinder itself. At each point on the circle, a line segment (the fiber) is attached vertically. The projection map $p: E \rightarrow B$ simply drops each point of the cylinder down to the corresponding point on the circle. This illustrates how the fiber is locally attached to the base space to form the total space. [8,15]

Homogeneous Spaces.

Let G be a Lie group and H a closed subgroup. The space G/H of left cosets is termed a homogeneous space, which is a manifold M where a Lie group has a transitive action. Consequently, M is diffeomorphic to the coset space G/H , with H being a closed Lie subgroup of G . This structure naturally induces a principal fiber bundle: $H \rightarrow G \rightarrow G/H$. [3]

In this context, we set $E = G$, $B = G/H$, and $F = H$.

Classifying Spaces.

In theory of fiber bundle of algebraic topology, a classifying space BG for a topological group G is a space whose entire topology is designed to encode the structure of *principal G -bundles*. [11]

Characteristic class:

"Characteristic classes are topological elements at cohomology classes that can be computed for a fiber bundle of base space mainly vector bundle." This sentence seems to be discussing characteristic classes in the context of topology and fiber bundles, but the wording is a bit unclear. Here's a clearer interpretation. [11,12]

Characteristic classes are algebraic invariants associated with vector bundles (likely meant by "vector bundle") over a base space. These

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classes live in the cohomology of the base space and provide topological information about the bundle. They are used to distinguish non-isomorphic bundles and play a key role in differential geometry and algebraic topology. For our goals, it is sufficient to focus on,

- (i) $Or(r) := Un(\mathcal{R}^r)$, or $O := Un(\mathcal{R}^\infty)$ the orthogonal – unitary groups
 - (ii) $SOr(r) := SUn(\mathcal{R}^r)$, or $SO :$
 $= SUn(\mathcal{R}^\infty)$ the specific orthogonal – unitary groups
 - (iii) $Un(r) := Un(\mathbb{C}^r)$, or $U := Un(\mathbb{C}^\infty)$ the unitary groups.
 - (iv) $SUn(r) := SUn(\mathbb{C}^r)$, or $SU :$
 $= SUn(\mathbb{C}^\infty)$ the specific unitary groups.
 - (v) $SP(r) := Un(\mathcal{H}^r)$, or $SP := Un(\mathcal{H}^\infty)$ the symplectic groups.
- [2]

In this algebraic topological context, we calculate the classes of features of several homogeneous areas, including these spaces

$$SP(r)/Un(r), Un(2r)/SP(r), Un(r)/Or(r), \text{ and } SOr(2r)/Un(r).$$

The Stiefel-Whitney, Euler, and Thom isomorphism classes, along with the Chern classes, are the initial elements of characteristic classes and some others we deal with in our study. Since there will be a lot of explicit calculations, using a few general notational conventions helps to maintain structure. Then the symbols of these classes denoted as,

$sw_i \in H^i(BOr(r); \mathbb{F}_2)$, denotes the classes of Stiefel Whitney.

$ch_i \in H^{2i}(BUn(r); \mathbb{Z})$, denotes the classes of Chern.

$sy_i \in H^{4i}(BSP(r); \mathbb{Z})$, denotes the classes of symplectic.

$P_i \in H^{4i}(BOr(r); \mathbb{Z})$, denotes the classes of Pontryagin .

$\mathcal{E} \in H^{2i}(BSOr(2r); \mathbb{Z})$, denotes the classes of Euler.[2,7]

Before inserting integrally, the P_i and $\mathcal{E}$ will be taken in mean of coefficient rings that contain half values. For fundamental illustrations of characteristic classes in cohomology with different factor rings, we employ same notations computing these

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rings of characteristic classes reduce to computing the cohomology rings $H^*(BUn(r); \mathbb{Z})$ and $H^*(BOr(r); \mathbb{Z}_2)$.

Definition 1. The quotient space D_λ/S_λ (S_λ collapses to a point) is defined the space of Thom by bundle λ and it's often referred as $Tm(\lambda)$ or $Th(\lambda)$.

We are going to make advantage of $Tm(\lambda)$ or Th_λ , then the global G -bundle's Thom space is represented by symbolled $ThmG$ or TmG , e.g., $ThUn(r), TmUn(r)$ or ThU, TmU like BU, BO , and other complex or real K -theory spaces, these basic Thom spaces show inequality and bordism. And the Thom space $Th(\mathcal{R}^n)$ computes as $Th(\mathcal{R}^r) = D^r/S^{r-1} = S^r \cong \mathcal{R}^r \cup \infty$. [5,9]

2- Classical groups:

The Lie classical groups, which are a category of leading groups, are now visible, for K denoted by real, complex and quadrated numbers $K = \mathcal{R}, \mathbb{C}$ and $\mathcal{H}$ respectively, we have linear translated of k with inner product of topological space $\mathcal{U}$ is; $Th : \mathcal{U} \rightarrow \mathcal{U}$ The classical groupings are thus as follows: [2]

$GL(\mathcal{U}) = \{Th : \mathcal{U} \rightarrow \mathcal{U} \mid Th \text{ is reversible and of finite type}\},$

$Un(\mathcal{U}) = \{Th \mid Th \in GL(\mathcal{U}) \text{ and } Th \text{ is an isometry}\},$

and, if $K = \mathcal{R}$ or $K = \mathbb{C}$

$SL(\mathcal{U}) = \{Th \mid Th \in GL(\mathcal{U}) : \det Th = 1\}$

$SU(\mathcal{U}) = \{Th \mid Th \in Un(\mathcal{U}) : \det Th = 1\}.$

The determinant is assuredly well-defined due to the finite type of demand. We can characterise this class of matrices by choosing a certain Orthonormal foundation for $\mathcal{U}$. It may be classified as a generic linear group, which functions as a topological group concerning matrices of the form. [4]

$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}$$

M is an inverted $r \times r$ matrix of values with $r < \infty$.

Consequently, the parts that are included $Un(\mathcal{U}) \hookrightarrow GL(\mathcal{U})$ and $SUn(\mathcal{U}) \hookrightarrow SL(\mathcal{U})$ these are the equivalences of homotopy

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Extra classical classifications are referred to as spinor groups, we define $\mathcal{S}Pin(r)$ for $r > 2$ with $\mathcal{S}Pin = \mathcal{S}Pin(\infty)$, as the complete covering group of $SO(r)$. Any one of the $\mathcal{S}Pin(r)$ for $r < \infty$ is a Lie group and every $\mathcal{S}Pin = \mathcal{S}Pin(r)$. Since $\pi_1(SO(r)) = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{S}Pin(r)$ is a double cover of $SO(r)$. Yielding the following identifications in low dimensions:

- (i) $SO(1) = SUn(1) = e$ and $Or(1) = \mathbb{Z}/2\mathbb{Z}$
- (ii) $SO(2) \cong Un(1) = Th(1)$ (the Thom circle group)
- (iii) $\mathcal{S}Pin(3) \cong SUn(2) \cong \mathcal{SP}(1) = S_3$ (Quaternary numbers of dimension one)
- (iv) $\mathcal{S}Pin(4) \cong \mathcal{SP}(1) \times \mathcal{SP}(1)$
- (v) $\mathcal{S}Pin(5) \cong \mathcal{SP}(2)$
- (vi) $\mathcal{S}Pin(6) \cong SUn(4)$. [2]

Proposition 1.[10] The trivial map $\pi: Y \rightarrow B$ of a main G -bundle is equivalent to the existence of a cross-section $\sigma: B \rightarrow Y$.

Proof. Necessity is obvious by given σ and an atlas $\{\tilde{V}_j, \varphi_j\}$, the maps $\psi_j: \tilde{V}_j \rightarrow G$ provided by $\psi_j(x) = \varphi_{j,x}^{-1} \circ \sigma(x)$, for $g \in G$, Satisfy

$$\psi_j(x)^{-1} g_{ji}(x) \psi_i(x) \in H = e, \text{ for } x \in \tilde{V}_i \cap \tilde{V}_j.$$

For computational purposes, the standard instances of Stiefel manifolds and their different bundles are recorded.

Lemma 1. For $1 \leq \delta \leq r$, $\tilde{V}_q(\mathcal{R}^r)$ is $(r - \delta - 1)$, connected and

$$\pi_{r-\delta}(\tilde{V}_\delta(\mathcal{R}^r)) = \begin{cases} \mathbb{Z} & < r, \delta = 1 \text{ or } r - \delta \text{ is even} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \delta = r \text{ or } r - \delta \text{ is odd} \end{cases}$$

Theorem 1. The $H^*(\tilde{V}_\delta(\mathcal{R}^r); \mathcal{F}_2)$ is cohomology, which takes an initial structure of equality $\{x_i \mid r - \delta \leq i < r\}$.

Proof. We start by inducing δ , starting with $\tilde{V}_1 = S^{r-1}$. Suppose the result is for $\tilde{V}_{\delta-1}$ and take note of the bundle's modulo 2 cohomology's Serre spectral sequence. $\tilde{V}_\delta \rightarrow \tilde{V}_{\delta-1}$ by fiber $S^{r-\delta}$. We have $E_2 = H^*(\tilde{V}_{\delta-1}; \mathcal{F}_2) \otimes H^*(S^{r-\delta}; \mathcal{F}_2)$ since $\pi_1(\tilde{V}_{\delta-1}) = 0$ if $q < n$ and since $\tilde{V}_r \rightarrow \tilde{V}_{r-1}$ is useless double cover $Or(r) \rightarrow SO(r)$. All differentials except $d_{r-\delta+1}$ are nulls dimensional reasons. If $d_{r-\delta+1}(\iota_{r-\delta})$ were non-zero, then $H^{r-\delta}(\tilde{V}_\delta; \mathcal{F}_2) = 0$. However, $H^{r-\delta}(\tilde{V}_\delta; \mathcal{F}_2) = \mathcal{F}_2$ by the Hurewicz and general coefficient theorems, as well as Lemma 1. [15]

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Therefore $E_2 = E_\infty$ and canonical maps are consulted in the conclusion.

$\iota : \mathbb{V}_{\delta-1}(\mathcal{R}^{r-1}) \rightarrow \mathbb{V}_\delta(\mathcal{R}^r)$ and $\pi : \mathbb{V}_{\delta+1}(\mathcal{R}^r) \rightarrow \mathbb{V}_\delta(\mathcal{R}^r)$

Lemma 2. For $K = \mathcal{R}, \mathbb{C}$, or $\mathcal{H}$, the greatest subgroup of $Un(K^{n-1})$ which is natural in $Un(K^r)$ is the narrow or null group.

Proof. is equivalent to the unit sphere S^{d_r-i} , $d = \dim_{\mathcal{R}} K$, then $Un(K^r)$ effective self-action appears by its.

Definition 2. The Stiefel manifold $\mathbb{V}_\delta(K^r)$ for $r \geq \delta > 0$, is the homogeneous space $Un(K)^r / Un(K^{r-1})$, so that, $\mathbb{V}_\delta$ is.[8]

$$\mathbb{V}_\delta(\mathcal{R}^r) = Or(r) / Or(r - \delta)$$

$$\mathbb{V}_\delta(\mathbb{C}^r) = Un(r) / Un(r - \delta)$$

$$\mathbb{V}_\delta(\mathcal{H}^r) = SP(r) / SP(r - \delta).$$

Lemma 3. For $r > \delta$, the fundamental mappings

$$SOr(r) / SOr(r - \delta) \rightarrow \mathbb{V}_\delta(\mathcal{R}^r).$$

$$\text{and } SUn(r) / SUn(r - \delta) \rightarrow \mathbb{V}_\delta(\mathbb{C}^r).$$

Classical homeomorphisms are discussed. Canonical homeomorphisms emerge.

$\mathbb{V}_r(\mathcal{R}^r)$		
$= Or(r)$	$\mathbb{V}_1(\mathbb{C}^r) = Un(r)$	$\mathbb{V}_1(\mathcal{H}^r)$
$\mathbb{V}_{r-1}(\mathcal{R}^r)$	$\mathbb{V}_{n-1}(\mathbb{C}^r) = SUn(r)$	$= SP(r)$
$= SOr(r)$	$\mathbb{V}_1(\mathbb{C}^r) = S_{r-1}$	$\mathbb{V}_1(\mathcal{H}^r) = S_{4r-1}.$
$\mathbb{V}_1(\mathcal{R}^r) = S_{r-1}$		

The following are the immediate effects of the previous results. [2,9]

If homology and cohomology are dealt with using integer coefficients, we define the Chern class, we may observe that the transgression is through the circle bundle of $S^1 = Un(1)$, that the transgression map $\tau: H^1(S^1) \rightarrow H^2(BUn(1))$ is an isomorphic structure; hence, the suspension is also

$$\sigma: H^2(BUn(1)) \rightarrow H^1(S^1).$$

Then Chern classes denoted as $ch_i \in H^{2i}(BUn(r))$ conclude the following fundamental facts. [12]

(i) $ch_0 = 1$ and $ch_i = 0$ if $i > r$.

(ii) $ch_1 \in H^2(BUn(1))$ is the normal canonical class.

(iii) $i_n^*(ch_i) = ch_i$ (hence $i_r^*(ch_r) = 0$).

$$(iv) p_{i,j}^*(ch_k) = \sum_{a+b=k} ch_a \otimes ch_b.$$

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If we define $H^*BSP(r)$ has comparable to the Chern classes in terms of polynomial generators of $H^*BUn(r)$ by canonical maps $\varphi_r: T^r \rightarrow SP(r)$, where φ_r is the induced map on classifying spaces was also presented by $\mu_1: SP(1) \rightarrow SUn(2)$ that forms topological groups as an isomorphism. Then the unique classes $sy_i \in H^{4i}BSp(r)$, these classes known as *simplistic classes*, obey the next axioms.[9]

(i) $sy_1 \in H^{4i}BSP(1)$ is c_2 under the identification $\mu_1: SP(1) \rightarrow \cong SUn(2)$.

(ii) $i_n^*(sy_i) = sy_i$.

(iii) $p_{i,j}^*(sy_i) = \sum_{a+b=l} sy_a \otimes sy_b$.

(iv) $sy_0 = 1$ and $sy_i = 0$ for $i > r$.

In addition, $H^*BSp(r) = P\{sy_1, \dots, sy_r\}$ and

(v) $\mu_r^*(ch_i) = 0$ if i is odd, and $\mu_n^*(ch_{2i}) = (-1)^i sy_i$.

(vi) $v_r^*(sy_i) = \sum_{a+b=2i} (-1)^{a+i} ch_a ch_b$.

Each characteristic class of the bundle $SP(r)$ is given as a polynomial in the symmetric well classes. If $\lambda: E \rightarrow A$ is an $SP(r)$ - bundle, under the symmetric well classes $sy_i(\lambda) \in H^{4i}(A)$ following:

(i) if $SP(1) \cong SUn(2)$ for bundle λ then $sy_1(\lambda) = -ch_2(\lambda^c)$.

(ii) $sy_i(\lambda) = sy_i(\lambda \oplus 1)$, where 1 indicates meaningless or a minor $SP(1)$ bundle associated with A. [11]

(iii) $sy(\lambda \oplus \eta) = sy(\lambda)sy(\eta)$, for each SP -bundle η .

(iv) $sy_0(\lambda) = 1$ and $sy_i(\lambda) = 0$ for $i > n$.

(v) $ch_i(\lambda^{ch}) = 0$ if i an odd case with $ch_{2i}(\lambda^{ch}) = (-1)^i sy_i$.

(vi) if λ is a $Un(n)bu$: $sy_i(\lambda_{\mathcal{H}}) = \sum_{a+b=2i} (-1)^{a+i} ch_a(\lambda) ch_b(\lambda)$.

Homogeneous specially spaces, $Un(2r)/SP(r)$, $SUn(2r)/SP(r)$, and

$S(r)/Un(r)$ can now be undertaken for the computation of the cohomology of homogeneous spaces.

Corollary 1. [2] In the form of algebras, $H^*Un(2r)/SP(r) = E\{x_{4i-3} \mid 1 \leq i \leq n\}$ and $H^*SUn(2r)/SP(r) = E\{x_{4i-3} \mid 2 \leq i \leq r\}$ with $|x_{4i-3}| = 4i - 3$. Then $H^*Un(2r)/SP(r) \rightarrow H^*Un(2r)$ and $H^*SUn(2r)/SP(r) \rightarrow H^*SUn(2r)$ take x_{4i-3} to x_{4i-3} .

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Proof. By taking spectral sequence associated with Eilenberg-Moore, $Un(2r)/SP(r) \rightarrow BSP(r) \rightarrow BUn(2r)$. We need to calculate

$E_2 = H^*BUn(2r) \text{ or } (\mathbb{Z}, H^*BSP(r))$, so let $H^*BUn(2r) = P\{ch_1, \dots, ch_{2r}\}$ and let $H^*BSP(r) = P\{sy_1, \dots, sy_r\}$, since E_2 is located in the second quadrant, and the fibrations map immediately supports this statement,

$$\begin{array}{ccccc} Un(2r) & \rightarrow & EUn(2r) & \rightarrow & BUn(2r) \\ \downarrow & & \downarrow & & \updownarrow \\ Un(2r)/SP(r) & \rightarrow & BSP(r) & \rightarrow & BUn(2r) \end{array}$$

"Because the suspensions of the Chern classes as generators for $H^*Un(2r)$, the calculation for $H^*SUn(2r)/SP(r)$ is precisely the same."

Corollary 2. As algebras,

$$H^*Un/SP = E\{x_1, x_5, x_9, \dots\},$$

$$H^*SUn/SP = E\{x_1, x_5, x_9, \dots\},$$

$$\text{and } H^*SP/Un = P\{ch_1, ch_2, \dots\}/I$$

where $I = \text{im } v^*$ is generated by

$$(\sum_{a+b=2i} (-1)^{a+i} ch_a ch_b \mid i \geq 1). [2]$$

Proof.[2] Instantaneous by passing to limits because the cohomological representations created by

$$Un(r)/SP(r) \rightarrow Un(r+1)/SP(r+1),$$

$$Un(r)/SP(r) \rightarrow SUn(r+1)/SP(r+1)$$

$$\text{and } SP(r)/Un(r) \rightarrow SP(r+1)/Un(r+1)$$

they are transformations maps that stand out. The homomorphisms that each of the maps in the bundles result in cohomology can now be calculated in summary, be.

$$\begin{array}{ccccc} Un & \xrightarrow{v} & SP & \xrightarrow{q} & Un \\ SP/Un & \xrightarrow{i} & BUn & \xrightarrow{v} & BSP \\ SP & \xrightarrow{\mu} & Un & \xrightarrow{p} & Un/SP \\ Un/SP & \xrightarrow{j} & BSP & \xrightarrow{\mu} & BUn \end{array}$$

Given that the finite case solutions follow naturally, we only consider the infinite situation.[2] If we deal with homology and cohomology with integer $2\mathbb{Z}$ coefficients, we define Stiefel-Whitney classes by

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remembering that $Or(1) = \mathbb{Z}/2\mathbb{Z}$ and then, $BOr(1) = K(\mathbb{Z}/2\mathbb{Z}, 1) \cong \mathbb{R}P^\infty$. Let $\varphi_r : BOr(1)^r \rightarrow BOr(r)$ be the induced map by: $Or(1)^r \subset Or(r)$ the inclusion case, by $ch_i \in H^{2i}(BUn(r); \mathbb{Z}/2\mathbb{Z})$, the modulo 2-reduction of $ch_i \in H^{2i}(BUn(r); \mathbb{Z})$ we denote the unique of Stiefel-Whitney classes as $sw_i \in H^i BOr(r)$, which satisfied by fallow theorem2.

Theorem 2:

- (i) $sw_0 = 1$ and $sw_i = 0$ if $i > n$.
- (ii) $sw_1 \in H^1(BOr(1))$
 $= H^1 \mathbb{R}P^\infty$ is the unique non – zero element.
- (iii) $i_r^*(sw_i) = sw_i$ (hence $i_r^*(sw_r) = 0$).
- (iv) $p_{i,j}^*(sw_k) = \sum_{a+b=k} sw_a \otimes sw_b$.

Moreover, $Hom^* BOr(r) = P\{sw_1, \dots, sw_r\}$ and, if we let $i_r^*(sw_i) = sw_i \in H^* BOr(r)$, then $H^* BOr(r) = P\{sw_2, \dots, sw_r\}$ and (with $sw_1 = 0$) properties (i) - (iii) remain valid.

Proof: [12]

In a series of phases, we shall demonstrate that $H^* BOr(r)$ is a polynomial algebra $P\{y_1, \dots, y_r\}$. The Stiefel-Whitney classes can then be formally defined using the methods from the unitary and symplectic cases. $H^* BOr(r) = P\{y_1, \dots, y_r\}$ with $\sigma^* y_i = x_{i-1}$ is implied by the coalgebra $H^* SOr(r) = E\{x_1, \dots, x_{r-1}\}$, which we know to exist. The map of fibrations can be used to demonstrate that the x_i are globally transgressive.

$$\begin{array}{ccccc} SOr(r) & \rightarrow & ESO(r) & \rightarrow & BSO(r) \\ \downarrow f_1 & & \downarrow F & & \updownarrow \\ SOr(r)/SO(r) & \rightarrow & BSO(r) & \rightarrow & BSO(r) \end{array}$$

i th Pontryagin class P_i in $H^* BOr(r)$ or $H^* BSO(r)$ is assigned as $(-1)^i v_r^*(ch_{2i})$. Note that $v_r^*(ch_i)$ is modulo 2-reduced to sw_i^2 by Theorem 2:(vi). Since w_{2i}^2 persevere to E_∞ relating to the Bockstein spectrum, $v_r^*(ch_{2i})$. has infinite order. Since $sw_{2i+1}^2 = \beta(sw_{2i}, sw_{2i+1})$, $v_r^*(ch_{2i+1})$ has order 2.

Theorem 3. The following characteristics apply to the integral

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Pontryagin classes: The λ - reduction of P_i is the essential classes P_i are distinguished by following states.

- (i) The modulo 2- reduction of P_i is sw_{2i}^2 .
- (ii) $i_r^*(P_i) = P_i$ and $j_r^*(P_i) = P_i$.
- (iii) $p_{ij}^*(P_k) \equiv \sum_{a+b=k} P_a \otimes P_b$ modulo 2-torsion.
- (iv) $v_r^*(ch_{2i}) = (-1)^i P_i$
- (v) $\mu_r^*(P_i) = \sum_{a+b=2i} (-1)^{a+i} ch_a ch_b$
- (vi) $H^*BOr(2r + \epsilon)$ and $H^*BSOr(2r + \epsilon)$, $\epsilon = 0$ or 1 , include $P\{P_1, \dots, P_r\}$ in the context of a subalgebra. [11,1,14]

The complexity of an $Or(r)$ -bundle is $\lambda_{\mathbb{C}}$ if the total Pontryagin class of that bundle is $P(\lambda) = 1 + P_1(\lambda) + P_2(\lambda) + \dots$. Let $\lambda_{\mathcal{R}}$ be the underlying $SOr(2r)$ bundle if λ is a $Un(r)$ -bundle that in terms of characteristic classes, we read theorem 3 as follows.

Corollary 3. *The Pontryagin classes fulfil: [12]*

- (i) The modulo 2- reduction of $P_i(\lambda)$ remains $(sw_{2i}(\lambda))^2$. The λ - reduction of $P_i(\lambda)$ remains $P_i(\lambda)$.
- (ii) $P_i(\lambda \oplus 1) = P_i(\lambda)$ if 1 is the minor lined bundle. If λ is $SOr(r)$ bundle and λ_* is η considered as $Or(r)$ bundle, then $P_i(\lambda) = P_i(\lambda_*)$.
- (iii) $P(\lambda \oplus \eta) \equiv P(\lambda)P(\eta)$ modulo 2 - torsion
- (iv) $ch_{2i}(\lambda_{\mathbb{C}}) = (-1)^i P_i(\lambda)$
- (v) $ch_{2i}(\lambda_{\mathbb{C}}) = (-1)^i P_i(\eta)$
- (vi) $P_i(\lambda_{\mathcal{R}}) = \sum_{a+b=2i} (-1)^{a+i} ch_a(\lambda) ch_b(\lambda)$

, or, equivalently, $1 - P_1(\lambda_{\mathcal{R}}) + P_2(\lambda_{\mathcal{R}}) - \dots = ch(\lambda)ch(\lambda)$.

Proof. $ch(\lambda \otimes \mathbb{C}) = ch(\lambda \oplus \bar{\lambda}) = ch(\lambda)ch(\bar{\lambda})$ For i odd the associated Chern class is $ch_i(\lambda \oplus \bar{\lambda}) = \sum_i (-1)^i ch_i(\lambda) ch_{k-i}(\lambda) = 0$. And so, product $ch(\lambda \otimes \mathbb{C})$ is simply the sum of even Chern classes $ch_{2i}(\lambda \otimes \mathbb{C})$. By the mean of Pontryagin classes this is $1 - p_1 + p_2 - \dots \pm p_r$. [12]

If we follow work modulo 2 - reduction, then $v_r^*(ch_{2i+1}) = sw_{2i+1}^2 = \beta(sw_{2i}sw_{2i+1})$. Therefore, $v_r^*(ch_{2i+1})$ has order 2 and so does $(ch_{2i+1})(\lambda_{\mathbb{C}})$.

The Euler class : $\mathcal{E} \in H^i BSOr(r)$ satisfies: [12,4,6]

- (i) The modulo 2- reduction of $\mathcal{E}$ is sw_r . The λ - reduction of $\mathcal{E}$ is $\mathcal{E}$ (which is null in $H^*BSOr(2r + 1)$). The Euler class $\mathcal{E}$ is defined by these

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two issues that $2\mathcal{E} = 0$ in $H^*B\text{SO}r(2r + 1)$, the $\mathcal{E}$ the infinite exhibits order within $\text{Hom}^*B\text{SO}r(2r)$

$$(ii) i_r^*(\mathcal{E}) = 0$$

$$(iii) p_{ij}^*(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}$$

$$(iv) v_r^*(ch_r) = (-1)^{[r/2]}\mathcal{E}^2$$

$$(v) \mu_r^*(\mathcal{E}) = ch_r(\mu_r: B\text{Un}(r) \rightarrow B\text{SO}r(2r))$$

$$(vi) \mathcal{E}^2 = P_r \text{ in } H^*B\text{SO}r(2r). H^*B\text{SO}r(2r) \text{ includes}$$

$P\{P_1, \dots, P_{r-1}, \mathcal{E}\}$ defining a subalgebra. $\text{Hom}^*B\text{SO}r(2r + 1)$ includes

$P\{P_1, \dots, P_r\} \otimes P_{\mathbb{Z}/2\mathbb{Z}}\{\mathcal{E}\}$ as a method involving a subalgebra.

(vii) $B\text{SO}r(r) \rightarrow B\text{SO}r(r)$ is outomorphism from the bundle $\text{Or}(r)/$

$S\text{Or}(r) = \mathbb{Z}/2\mathbb{Z} \rightarrow B\text{SO}r(r) \rightarrow B\text{Or}(r)$ sends $\mathcal{E}$ to $-\mathcal{E}$.

Corollary 4. A Euler class $\mathcal{E}$ of the $S\text{Or}(r)$ –bundle has axioms:

(i) The modulo2- reduction of $\mathcal{E}(\lambda)$ is $sw_r(\lambda)$. The λ -reduction of $\mathcal{E}(\lambda)$ is $\mathcal{E}(\lambda)$. If λ is odd case in its dimensional : $2\mathcal{E}(\lambda) = 0$.

(ii) $\lambda(\lambda \oplus 1) = 0$ where 1 denotes trivial or a minor lined bundle.

(iii) $\lambda(\lambda \oplus \eta) = \mathcal{E}(\lambda)\mathcal{E}(\eta)$.

(iv) If λ is an $S\text{Or}(r)$ -bundle : $c_r(\lambda_{\mathbb{C}}) = (-1)^{[r/2]}\mathcal{E}(\lambda)^2$.

(v) If λ is a $\text{Un}(r)$ - bundle : $\mathcal{E}(\lambda_{\mathbb{R}}) = ch_r(\lambda)$.

(vi) If λ is an $S\text{Or}(2r)$ -bundle : $\mathcal{E}(\lambda)^2 = P_r(\lambda)$. [10,12,13]

(vii) If λ denotes $\tilde{\lambda}$ to be opposite orientable : $\mathcal{E}(\tilde{\lambda}) = -\mathcal{E}(\lambda)$.

The two most significant classes of integral characteristics are the Euler and Pontryagin classes. They cover every class that is devoid of torsion.

A family of torsion classes is now under consideration. In

$\text{Hom}^*B\text{SO}r(2r); \mathbb{Z}/2\mathbb{Z}$ we have only $\beta(sw_{2i}) = sw_{2i+1}$, for this in $\text{Hom}^*B\text{SO}r(r)$, another class of order 2 is present, reduces modulo 2 to sw_{2i+1} .

Conclusion:

This study has demonstrated deep interrelationships among several well-known characteristic classes and how each influences the evolution of the others in solving classical problems in algebraic topology. So, this study improves comprehension of essential ideas in algebraic topology and geometry of homogenous spaces by examining the calculations and algebraic relationships. These relationships enable the systematic calculation of characteristic classes for diverse bundles in homogeneous spaces. We have demonstrated that Pontryagin classes in case of real bundles and Chern classes for complex bundles are fundamentally interconnected, with the Euler class representing the highest Chern class in oriented bundles, its square being directly

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related to the top Pontryagin class. These interconnections reveal substantial algebraic relationships that aid in the computation of characteristic classes and deepen our understanding of the emergence of topological properties from the geometry of bundles. Using these connections gives researchers better computational tools and a clearer way to solve problems in modern topology [6, 11], also by looking at these connections more up close, in a focused manner, researchers can learn more about how vector bundles work and how they are used in mathematical physics. Lastly, these ongoing conversations between algebraic and topological methods help us understand the complex networks that support mathematical physics theories.

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