

FULL PAPER**Weakly Continuity of Bounded Functions in Certain Banach Spaces***Prepared by*

Habeeb Ibrahim
Department of Mathematics
College of Science
Qassim University
Buraydah 51452 -Saudi Arabia
Ha.Ibrahim@qu.edu.sa

Abstract.

We defined Rosenthal Space, and gave description of some Banach spaces those are not containing the space l_1 in terms of holomorphic continuous mappings. The polynomials those are continuous weakly sequentially type, and bounded subsets those are weakly continuous type has been described. Finally, we gave characterizations of bounded linear operators.

Key words: bounded linear, Rosenthal space, symmetric operator, probability measure

1. Introduction: Through, we consider A and B are Banach spaces. We write A^* and F_A to signify the dual of A the closed unit ball of A respectively. The sets $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote natural, the real, and the complex numbers respectively. Suppose that $C_{wb}(A, B)$ signifies the space of all functions convert the sequences in A that are weakly convergent to convergent ones in B , and $C_{wj}(A, B)$ signifies the space of functions those convert the bounded subsets of A to weakly continuous. It is evident that, $C_{wj}(A, B) \subseteq C_{wb}(A, B)$. Let A and B be complex Banach spaces, then, the space $\mathcal{H}(A, B)$ is known as a space of all functions of holomorphic type from A to B .

For each natural number $j \in \mathbb{N}$, $\mathcal{P}(A_k, B)$ denotes the space of polynomials from A to B those are k -homogeneous continuous type. We identify that $\mathcal{P}(A_0, B) = B$. $L_s(A_j, B)$ denotes the space of k -linear functions from $A \times \dots \times A$ to B those are

continuous and symmetric. Let $L_s(A_j, B) \rightarrow \mathcal{P}(A_j, B)$ be an operator transforms the k -linear function E into the polynomial P which defined by $P(y) = E(y, \dots, y)$ is called an isomorphism of Banach spaces [15]. Assume that $\psi(A, B)$ denotes the continuous maps family from A to B , then we can write:

$$\phi_\beta(A, B) = \phi(A, B) \cap C_\beta(A, B)$$

for $\beta = wj$ or wb . Throughout, if we omitted the range space it is clear that, $\mathbb{K} = (\mathbb{C} \text{ or } \mathbb{R})$; and thus, $\mathcal{P}(A_j) = \mathcal{P}(A_j, \mathbb{K})$.

Such function spaces have been examined by many authors ([2, 4,5,6,7, 9, 11, 13]) in recent years. This note aims to provide revisions of results from [6,10] that characterize Banach spaces that do not contain l_1 in terms of the spaces discussed above.

If $S: A \rightarrow B$ is a bounded linear operator which transforms the weakly convergent sequences in A into convergent ones in B , then it is said to be completely continuous. It is evident that $\mathcal{P}_{wb}(A_1, B)$ represents the space of operators from A to B that is completely continuous. However, according to Proposition 2.5 of [7], $\mathcal{P}_{wj}(A_1, B)$ corresponds to the compact operator space from A to B . If every sequence in X admits of a Cauchy subsequence of weakly type, then the set $A \subset X$ is called a conditionally weakly compact subset or Rosenthal subset.

2. Basic Definitions and main concepts:

Definition 2.1: The Rosenthal space is the first complemented subspace of L_p that has been identified. It is the closed linear span in L_p of an independent three-valued symmetric sequence $(g_n)_{n=1}^\infty$ such that, if $q > 2$, $(\|g_n\|_2 / \|g_n\|_q)_{n=1}^\infty$ diminishes slowly.

Definition 2.2: A function $g: X \rightarrow B$ is considered weakly continuous if, for every $y \in X$ and $\varepsilon > 0$, there are $\psi_1, \dots, \psi_n \in A^1$, $\delta > 0 : x \in X$, $|\psi_i(y - x)| < \delta \forall i = 1, \dots, n$, then $\|g(y) - g(x)\| < \varepsilon$.

Definition 2.3:

An operator $S: A \rightarrow A^*$ is said to be a symmetric operator if S verifies:

$$\langle x, Sy \rangle = \langle y, Sx \rangle, \quad (y, x \in A).$$

Theorem 2.4. Each Rosenthal subset in a Banach space has a weak closure that is weakly sequentially dense.

Corollary 2.5. The functions g_i 's where $g_i: A \rightarrow B$ whose limitations to Rosenthal subsets of A are continuous and weakly comprise the space $C_{wb}(A, B)$.

Proof. Let $g \in C_{wb}(A, B)$ and $X \subset A$ be a Rosenthal subset. Theorem (2.4) states that there is $(x_n) \subset X$ such that $x_n \rightarrow x$ weakly for any $x \in \bar{X}^{\sigma(A, A^*)}$. Because $(g(x_n))$ converges to $g(x)$. Hence $g(\bar{X}^{\sigma(A, A^*)}) \subset \overline{g(X)}$, and $g|_X$ is weakly continuous.

Theorem 2.6: let A signifies a Banach space of complex type. Then the statements that follow are equivalent:

- (i) There is no copy of the space l_1 in A .
- (ii) For every B , the two spaces $C_{wb}(A, B), C_{wj}(A, B)$ are equal.
- (iii) For every complex B , the two spaces $\mathcal{H}_{wb}(A, B), \mathcal{H}_{wj}(A, B)$ are equal.
- (iv) For some complex B , the two spaces $\mathcal{H}_{wb}(A, B), \mathcal{H}_{wj}(A, B)$ are equal.
- (v) The two spaces $\mathcal{H}_{wb}(A), \mathcal{H}_{wj}(A)$ are equal.

Proof: (i) $\Rightarrow$ (ii) Since the bounded subsets are Rosenthal in a space that does not contain l_1 , then Corollary 2.5 is sufficient.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are evident.

(iv) $\Rightarrow$ (v): Assume that $g \in \mathcal{H}_{wb}(A, B) \setminus \mathcal{H}_{wj}(A, B)$. let $k: \mathbb{R} \rightarrow B$ be given by $k(\lambda) = \lambda(x)$ and choose $x \in g$ with $\|x\| = 1$. Then $k \circ g \in \mathcal{H}_{wb}(A, B) \setminus \mathcal{H}_{wj}(A, B)$.

(v) $\Rightarrow$ (i) Let S signifies the natural inclusion defined as $S: l_1 \rightarrow c_0$, T is an isomorphism defined as $T: M \rightarrow l_1$ and $M \subseteq A$ be a closed subspace. S can be extended to an operator V by using standard arguments to factor across $L_\infty[0, I]: A \rightarrow c_0$. V is totally continuous since $L_\infty[0, I]$ possesses the Dunford-Pettis property and all operators from the space $L_\infty[0, I]$ to c_0 are weakly compact. If (e_n) represents l_1 's unit vector basis, then write $\psi_n = 2e_n \circ U$. The sequence $\psi_n \subset A^*$ is obviously $\sigma(A^*, A)$ -null. Now define a function $g: A \rightarrow \mathbb{C}$ by:

$$g(y) = \sum_{n=1}^{\infty} (\psi_n(y))^n, \quad \text{where } (y \in A).$$

Then g is holomorphic and clearly characterized [15, 5.5].

We now demonstrate that $g \notin C_{wj}(A)$. In fact, if not, we would have that: $f: g \circ T^{-1} g \in C_{wj}(l_1)$. The n th coordinate mapping on l_1 is indicated by p_n , so:

$$\psi_n \circ T^{-1}(x) = 2e_n \circ V \circ T^{-1}(x) = 2e_n \circ S(x) = 2p_n(x), (x \in l_1)$$

so, for $x \in l_1$

$$f(x) = g \circ T^{-1}(x) = \sum_{n=1}^{\infty} (\psi_n \circ T^{-1}(x))^n = \sum_{n=1}^{\infty} (2p_n(x))^n.$$

We now demonstrate that f is not weakly continuous on the 2-ball. This streamlines the author's initial evidence and is based on a concept proposed by Professor R. M. Aron. let $\{\eta^1, \dots, \eta^k\} \subset L_{\infty}$ where $\eta^1 = (\eta_n^k)_{n=1}^{\infty}$ ($1 \leq k \leq j$). We can find an integer number $r \in \mathbb{Z}$ such that, for $q > p \geq r$, we can discover a sequence $(n_i) \subset \mathbb{N}$ that is an increasing sequence, where for each k ($1 \leq k \leq j$), the subsequence $(\eta_{n_i}^k)_i$ is convergent; and we obtain:

$$|\eta_{n_q}^k - \eta_{n_r}^k| \leq 1 + \varepsilon, \quad (1 \leq k \leq j)$$

For $x = e_{n_q} - e_{n_r}$ in the space $\in l_1$, with $q > r$, we get:

$$\|x\| = 2, |\eta^k(x)| = |\eta_{n_q}^k - \eta_{n_r}^k| \leq 1 + \varepsilon, \quad (1 \leq k \leq j),$$

and

$$f(x) = q + 2^{n_r}(-1)^{nr} \geq 2(1 + \varepsilon).$$

Lastly, we demonstrate that $g \in C_{wb}(A)$. In fact, let $L \subseteq A$ be weakly compact set. Then the set $V(L)$ easily be compact in the c_0 where $\varepsilon > 0$ and $j \in \mathbb{N}$ with $2^j < \varepsilon$. Since $(e_n) \rightarrow 0$ uniformly over compact subsets of the set c_0 due to its weak-star nullity, we find a positive integer $m \in \mathbb{N}$ ($m \geq j$) such that:

$$|2e_n \circ V(y)| \leq \frac{1}{4}(1 + \varepsilon),$$

where $y \in L$ and $n \geq m$.

and so

$$|\psi_n(y)| < 2^{-n},$$

where $y \in L$ and n is greater than or equal m .

For every $N \in \mathbb{N}$, let $g_N(y) = \sum_{n=1}^N (\psi_n(y))^n$. It is evident that $g_N \in C_{wb}(A)$. Suppose that $N > M > m$. Then for every $y \in L$ we obtain,

$$|g_N(y) - g_M(y)| \leq \sum_{n=1}^N |\psi_n(y)|^n < \sum_{n=M+1}^{\infty} 2^{-n} = 2^{-M} < 2^{-j} < 1 + \varepsilon.$$

Consequently, $(g_N)_{N=1}^{\infty} \subset C_{wk}(A)$ is a Cauchy sequence in the topology of uniform convergence on weakly compact subsets of A . Since $C_{wb}(A)$ is the completion of $C_{wj}(A)$ in this topology [10, Proposition 2], we can deduce that $g \in C_{wb}(A)$ and $(f_N)_{N=1}^{\infty}$ converges to f . The equivalence (a) $\Leftrightarrow$ (b) was proved in [10].

3.Main Results:

Theorem 3.1. The following statements are equivalent.

- (i) There is no copy of l_1 in A .
- (ii) For each B and $j \in \mathbb{N}$, $\mathcal{P}_{wb}(A_b, B) = \mathcal{P}_{wj}(A_b, B)$.
- (iii) There is B such that, for each $j \in \mathbb{N}$, $\mathcal{P}_{wb}(A_j, B) = \mathcal{P}_{wj}(A_b, B)$.
- (iv) For some B and $j \in \mathbb{N}$ ($j \geq 2$), $\mathcal{P}_{wb}(A_b, B) = \mathcal{P}_{wj}(A_b, B)$.
- (v) For some $j \in \mathbb{N}$ ($j \geq 2$), $\mathcal{P}_{wb}(A_b) = \mathcal{P}_{wj}(A_b)$.

Proof. (i) $\Rightarrow$ (ii): Applying Theorem 2.5 (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are evident.

(iv) $\Rightarrow$ (v) as in Theorem 2.6.

(v) $\Rightarrow$ (i) Assume that $T: M \rightarrow l_1$ is an isomorphism and $M \subset A$ is subspace of closed type. Assume that S from l_1 to l_2 is the natural inclusion map. Given that S is absolutely summing [13], the theorem of Grothendieck-Pietsch domination can be used [9]. Then, S factors through the space $L^\infty(\mu)$ for a regular probability measure μ of Borel, that is specified on a compact space. Because $L^\infty(\mu)$ is injective, T extends to the operator $U: A \rightarrow l_2$, that is completely continuous due to $L^\infty(\mu)$'s Dunford-Pettis property. For each y in A , write $U(y) = (U_n(y))_{n=1}^{\infty} \in l_2$. For every positive integer $j \geq 2$, we define $P_j: A \rightarrow \mathbb{N}$ by

$$P_j(y): \sum_{n=1}^{\infty} (U_n(y))^{j+\varepsilon}, \varepsilon > 0$$

P_j is the following functions' product:

$$A \xrightarrow{u} l_2 \xrightarrow{I} l_2 \xrightarrow{W} J,$$

where W is the polynomial of j -homogenous type and I is the natural inclusion given by:

$$W(\eta): \sum_{n=1}^{\infty} (\eta_n)^{j+\varepsilon} \text{ for } \eta = (\eta_n) \in l_j, \varepsilon > 0$$

Hence, $P_j \in \mathcal{P}(A_j)$. Additionally, because U is entirely continuous, $P_j \in \mathcal{P}_{wj}(A_j)$.

Now we are going to see that $P_j \notin \mathcal{P}_{wj}(A_b)$. If not, we would have

$$R_j = P_j \circ T^{-1}(x) \in \mathcal{P}_{wj}(l_{1b}).$$

Let p_n signifies the n th coordinate function on the space l_1 , we have:

$$R_j(x) = P_j \circ T^{-1}(x) \leq \sum_{n=1}^{\infty} (U_n \circ T^{-1}(x))^{j+\varepsilon} \leq \sum_{n=1}^{\infty} (p_n(x))^{j+\varepsilon}, (x \in l_1)$$

The j -linear function X_j of continuous symmetric type that is associated to R_j is given by:

$$X_j(x_1, \dots, x_j) = \sum_{n=1}^{\infty} p_n(x_1) \cdot p_n(x_2) \cdot \dots \cdot p_n(x_j), (x_1, \dots, x_j \in l_1).$$

We define operator C_j from the space l_1 to the space $L_s(l_{1+j+\varepsilon-1})$ as follows:

$$C_j(y)(x_1, \dots, x_{j-1}) = X_j(y, x_1, \dots, x_{j-1}), (y, x_1, \dots, x_{j-1} \in l_1)$$

Suppose that $(e_n) \in l_1$ where $n \neq m$ be a unit vector basis, then we have,

$$\begin{aligned} \|C_j(e_n) - C_j(e_m)\| &\geq |C_j(e_n)(e_n, e_n, \dots, e_n)^{(j-1)}, e_n) - C_j(e_m)(e_n, e_n, \dots, e_n)^{(j-1)}, e_n)| \\ &= |X_j(e_n, e_n, \dots, e_n)^j, e_n) - X_j(e_m, e_n, \dots, e_n)^{j-1}, e_n)| = 1, \end{aligned}$$

Since C_j is not compact, we may use [6, Theorem 2.9] to determine that

$$R_j \notin \mathcal{P}_{wj}(l_{1b}).$$

Another demonstration of Theorem 2.6 on holomorphic functions might be provided using the same reasoning. However, we offer both because Theorem 3.1 provides an actual illustration of a holomorphic function that may be helpful in the other applications and only requires fundamental tools from Banach space theory.

According to Proposition 2.12 of [6], the equivalency (i) $\Leftrightarrow$ (ii) is demonstrated. Known that (iii), (iv), and (v) are new.

The last theorem's assertions (iv) and (v) demonstrate how operators and polynomials behave differently. Odell demonstrated that if and only if every operator on A that is completely continuous is compact, then A does not include a duplicate of the space l_1 [16]. However, since $\mathcal{P}_{wb}(A_1) = \mathcal{P}_{wj}(A_1) = A^*$ for all A , Theorem 2.6 is no longer valid for $j = 1$ in (v). The theorem would not hold for $j = 1$ even if A were constrained in (iv) to be infinite dimensional. In fact, it is known from Proposition 3.7 of [8] that A does not include a complemented copy of the space l_1 if and only if there is an infinite-dimensional B where all operators that are entirely continuous from A to B are compact. Lastly, we provide an operator corollary.

In [3, 8], the symmetric operators are examined in connection with Arens regularity and spectral features of algebras of analytic functions. Whether every symmetric operator from A to A^* is weakly compact implies that every operator from A to A^* is also weakly compact is an unresolved question. Here, we address a related query.

Corollary 3.2: The following statements are equivalent.

(i) A has no copies of the space l_1 .

- (ii) All operators on A that are completely continuous are compact.
 (iii) All operators from A to A^* that are completely continuous are compact.
 (iv) All operators from A to A^* that are symmetric and completely continuous are compact.

Proof, (i) $\Rightarrow$ (ii) is included in (i) $\Rightarrow$ (ii) of Theorem 2.6.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are evident.

(iv) $\Rightarrow$ (i) Assume that there is an isomorphic duplicate of l_1 in A . We can locate a completely continuous operator, just as in the proof of Theorem 3.1. $U: A \rightarrow l_2$ expanding the inclusion $S: l_1 \rightarrow l_2$. Let $U(y) = (U_n(y))_{n=1}^{\infty} \in l_2$ For each $y \in A$. We define the polynomial $P: A \rightarrow \mathbb{K}$ of 2-homogeneous type by:

$$P(y) = \sum_{n=1}^{\infty} (U_n(y))^2.$$

Assume that $C: A \rightarrow A^*$ represents the corresponding operator given as follows:

$$\langle y, C(y) \rangle = \sum_{n=1}^{\infty} U_n(y) \cdot U_n(x), (y, x \in A).$$

According to Theorem 2.9 of [6] and since $P \notin \mathcal{P}_{wj}(A_2)$, C is not compact. C is clearly symmetric, and if $(y_m)_{m=1}^{\infty} \subset A$ is a sequence that is weakly and null, then:

$$\begin{aligned} \|C(y_m)\| &= \sup\{|\langle x, C(y_m) \rangle| : x \in Y_A\} = \sup\{|\langle U(y_m), U(x) \rangle| : x \in Y_A\} \\ &\leq \|U(y_m)\| \cdot \|U\|. \end{aligned}$$

Since U is an operator that is completely and continuous, so is C .

The equivalence (i) $\Leftrightarrow$ (ii) is hold from Odell's theorem [16].

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