ISSN (Print) **2794-7629** **Received** **11/03/2025**
ISSN (Online) **2794- 4549** **Accepted** **20/04/2025**

FULL PAPER

Survey on Fixed Point Theorem in Some Metric Spaces

Prepared by

Mohammed Nour A. Rabih
Department of Mathematics
College of science
Qassim university
Buraydah 52571,Saudi Arabia
M.Fadlallah@qu.edu.sa

Abstract

The aim of this article is to concisely consolidate and present a diverse array of works that are relevant to theory of fixed-point for specific metric spaces. This compendium may be advantageous to emerging researchers in the field of functional and nonlinear analysis. Within a metric space that is entirely completed, we show that the iteration of $x = f(x)$ is the point at which $f(x)$ converges. We present a number of fixed-point theorems for mappings that satisfy specific contractionary conditions that are contingent upon the fixed point.

Key Words: theory of fixed points, Banach contraction, metric spacesm, con metric spaces, complex valued metric spaces

1. Introduction

One of the most important topics in the study of metric spaces is fixed point theory. Whether metric spaces actually offer enough room for this theory is one of the important problems that can come up in this situation. Huang and Zhang recently proposed that the answer is negative in [27]. In fact, Huang and Zhang were the ones who first invented the cone metric space. Additionally, they looked at the existence and uniqueness of the fixed point of a self-map on a cone metric space.

Throughout the evolution of nonlinear analysis, the Banach contraction principle is a fundamental component that is widely recognized [1].This conclusion can be expressed as expressed in the theorem. This principle was expanded and enhanced in numerous ways, resulting in the development of a variety of fixed-point theorems. There are two conventional methods for enhancing and extending the Banach contraction principle:

1. Expanding the contraction condition of the theorem 2.3 to more universal contraction conditions
2. To change the comprehensive metric space (X, d) utilizing particular generalized metric spaces.

Edelstein [20, 21], Rakotch [45], Kannan [33], Bianchini [13], and a few others were responsible for the initial results in the first aforementioned direction. Rhoades' renowned research paper [47] and the subsequent paper by Collaço, P. and Silva, J.C.E. [15] both compiled and compared related results. The Banach contraction principle is typically demonstrated by the authors as a straightforward consequence of their result. This is done to illustrate that the Banach contraction principle is a suitable generalization of the fixed point theorem. For further details on these counterexamples, please refer to the proofs of [47, Theorem 1] and [15, Section 3].

Numerous novel generalized metric spaces were generated in the second direction by altering the metric axioms. For more details, see [24, 39, 41, 27]. Authors typically demonstrate that each metric is a generalized metric for a generalized metric and that there is a generalized metric that is not a metric on the same set. However, the conditions for contraction may remain constant when specific modifications are implemented, and the resulting generalized metric space commonly seems to be metrizable.

The results of fixed points in certain generalisations of metric spaces are capable of being generated from those in canonical metric spaces, as demonstrated by recent research conducted by a number of authors [6, 19, 18, 22, 26, 30, 31, 36]. It is crucial to recognize that not all of the methods mentioned result in trivial extended fixed-point results. In particular, these findings are contingent upon specific contraction conditions, and it is not always evident whether or not these conditions remain valid when the problem is examined in the corresponding metric space, for more details, we refer the reader to references [18, 19, 22, 26, 31, 36].

New types of generalized metric spaces have emerged in recent years, and many of these spaces have been thought of as hybrids of the earlier types. These include D^*-metric spaces [52] by Sedghi, cone rectangular metric spaces [8] by Azam, tvs-cone metric spaces [11,32] by Beg, I., Abbas, and Kadelburg, extended cone metric spaces or $tvs-G$ cone metric spaces [10] by Beg, I., Abbas, and Kadelburg, metric-type spaces [35,34] by Khamsi and Hussain, cv- metric spaces [9] by Azam, PC- metric spaces [38] by Malhotra, and cone symmetric spaces. [44] Partial-G-metric spaces, Radenović, and GP-metric spaces [50,58] S-metric spaces by Salimi and Zand [51] Amini-Harandi's metric-like spaces [5], Shah's quasi-b-metric spaces [53], Roldán's G^*-metric spaces [48], Popović's tvs-cone b-metric spaces [42], Shukla's partial rectangular metric spaces [55], and others Alghamdi, g-3ps; Radenović, G_b-metric spaces [17] Ding, b-metric-like spaces [4] ,[14] by Chaipunya, non-Archimedean cone metric spaces [29] by Hussain, and G_{Pb}-metric spaces [28]. Huy, Mukheimer's complex valued b-metric spaces [40], Zhu's quasi-metric-like spaces [59], Asadi's M-metric spaces [7], and Ahmed, A.E.S., and Shukla's qv- metric spaces [3]. provided pb-metric spaces, among other things [54]. The main method for creating metric spaces or certain generalized metric spaces is to undermine their axioms. The presentations of these papers are generally consistent; the authors introduce the concept of an extended metric space and subsequently state several theorems of fixed points in such spaces. The topological characteristics of new extended metric spaces are not always evident, nor is the relationship between theorems of fixed point in these new generalized metric spaces and those in older generalized metric spaces always obvious.

In this paper, the focus is on the fixed-point theory in metric spaces, cone metric spaces, and complex value metric spaces. The purpose of this survey is to provide a resource for graduate and undergraduate students studying the fixed-point theory in these metric spaces. To learn more about the relationship between fixed-point theories in different metric spaces, we refer the reader to [22], [31], and [43].

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2. Preliminaries

In the previous section, Introduction, we explained the importance of the fixed point in metric spaces and its development. In this section, we discuss some basic definitions and concepts related to metric spaces, the contraction application, and the fixed point.in the development of nonlinear analysis and fixed point theory, the Banach contraction principle is a renowned cornerstone.

Definition 2.1. Suppose that a pair (X, d) is a metric space. And a mapping $\mathcal{T}: X \to X$ is contraction mapping, that is available a constant c: $0 \leq c < 1$, indicating that

$$d\big(\mathcal{T}(x), \mathcal{T}(y)\big) \leq cd(x, y), \qquad \forall\, x, y \in X. \#(1)$$

Therefore, a contraction represents elements that are more closely spaced. Particularly, for each $x \in X$ and any $r > 0$, all points y in a ball $B_r(x)$ are mapped into a ball $B_s(\mathcal{T}x)$, where s is less than r. At times, a contraction is also referred to as a map that holds for (1) with $c = 1$, whereas a strict contraction is a map that satisfies (1) with $c < 1$. It is clear from (1) that a mapping is a contraction uniformly continuous.

In X, there is a point x.such that when $\mathcal{T}: X \to X$,

$$x = \mathcal{T}(x) \#(2)$$

This is what we call a fixed point for $\mathcal{T}$. A rigorous contraction on a complete metric space has a fixed point that is a unique, according to the contraction mapping theorem.

Theorem 2.2. ("Contraction mapping") For a mapping $\mathcal{T}: X \to X$ when $\mathcal{T}$ is a contraction on a complete metric space (X, d), there is a unique solution $x \in X$ to the equation (2).

Proof. As it will explicitly generate a sequence that converges to the fixed point, the proof is constructive in character. Allow x_0 to be any point in X. In X, a sequence (x_j) is defined as

$$x_{j+1} = \mathcal{T}x_j \quad \text{for } j \geq 0$$

The parentheses surrounding the argument of a map are frequently omitted in order to simplify the notation. The jth iterate of $\mathcal{T}$ is denoted as $\mathcal{T}^j$, and x_j is equivalent to $\mathcal{T}^j x_0$.

First, we demonstrate that (x_j) is a Cauchy sequence. From (1) and the triangle inequality, it follows that if $j \geq i \geq 1$,

$$\begin{aligned} d(x_j, x_i) &= d(\mathcal{T}^j x_0, \mathcal{T}^i x_0) \\ &\leq c^i d(\mathcal{T}^{j-i} x_0, x_0) \\ &\leq c^i [d(\mathcal{T}^{j-i} x_0, \mathcal{T}^{j-i-1} x_0) + d(\mathcal{T}^{j-i-1} x_0, \mathcal{T}^{j-i-2} x_0) + \cdots + d(\mathcal{T}x_0, x_0)] \\ &\leq c^i [\textstyle\sum_{k=0}^{j-i-1} c^k] d(x_1, x_0) \end{aligned}$$

$$\leq c^i[\sum_{k=0}^{\infty} c^k]d(x_1, x_0)$$

$$\leq (\frac{c^i}{1-c})d(x_1, x_0)$$

which indicates that the value (x_j) is Cauchy. (x_j) approach to a limit $x \in X$, as X is complete. The fact that the limit x is a fixed point of $\mathcal{T}$ is a consequence of $\mathcal{T}$ being continuous:

$$\mathcal{T}x = \mathcal{T}\lim_{j\to\infty} x_j = \lim_{j\to\infty} \mathcal{T}x_j = \lim_{j\to\infty} x_{j+1} = x.$$

Lastly, if x and y are two set points, then

$$0 \leq d(x, y) = d(\mathcal{T}x, \mathcal{T}y) \leq cd(x, y).$$

We have $d(x, y) = 0$ because $c < 1$, it indicates that $x = y$ and we have a unique fixed point.

Theorem 2.3. Under the assumption that a pair(X, d) is a metric space that is entirely complete and $\mathcal{T}$ is a self-map on X. Then

$$d(\mathcal{T}x, \mathcal{T}y) \leq cd(x, y)\#(3)$$

for all $x, y \in X$ and a value of $c \in [0,1)$. Then $\mathcal{T}$ has a unique fixed point $x^* \in X$, and $\lim_{j\to\infty} \mathcal{T}^{jx} = x^*$ for all $x \in X$.

As indicated in the introduction, this principle has been expanded and enhanced in numerous ways, resulting in the development of various fixed-point theorems. The Banach contraction principle can be improved and made more general in two ways: either the complete metric space (X, d) is replaced with a generalized metric space or the contraction condition (3) is changed to more general contraction conditions.

3. Fixed point in Metric Space

Assume that the metric space (X, d) is complete. To indicate the "zero" of X, we select an arbitrary $x_0 \in X$.

$$d(x, x_0) = \|x\|, \qquad \forall x \in X$$

In this study, the function $\varphi: [0,1] \to [0, \infty)$ An designated increasing function that vanishes with continuity at zero. In the event that α is greater than 1, we will also evaluate the (vanishing) sequence.

$$\omega_j(\alpha) = \left(\frac{\alpha}{j}\right)^{\alpha} \sum_{\kappa=1}^{j} \varphi\left(\frac{\alpha}{\kappa}\right)$$

The following theorem is true for a function $f: X \to X$.

Theorem 3.1. Assume that $\lambda \geq 0, \alpha \geq 1$ and $\beta \in [0, \alpha]$ be constants. If the following inequality

$$d(f(x), f(y)) \leq (1 - \epsilon)d(x, y) + \lambda\epsilon^{\alpha}\varphi(\epsilon)[1 + \|x\| + \|y\|]^{\beta}\#(4)$$

is true $\forall\, \epsilon \in [0,1]$ and every $x, y \in X$, Consequently, f has fixed point that is unique, f takes a singular fixed point $x_\star = f(x_\star)$. Additionally, referring to $f^j = f \circ \cdots \circ f$ (j times),

$$d\left(x_\star, f^j(x_0)\right) \leq c\omega_j(\alpha) \#(5)$$

for a positive constant c when $c \leq \lambda(1+4\|x_\star\|)^\beta$.

Corollary 3.2. The convergence rate estimate (5) is obtained by setting $\Theta = \alpha + \gamma$ if $\varphi(\epsilon) = \epsilon^\gamma$ for some $\gamma > 0$.

$$d\left(x_\star, f^j(x_0)\right) \leq \frac{c\Theta^\Theta}{j^{\Theta-1}} \#(6)$$

Remark.3.3. The validity of (1) is plainly independent of the specific choice of the zero of X, up to the point of redefining λ, so selecting x_0 instead of a generic $x \in X$ is not restrictive.

Corollary 3.4. In the case that a contraction of f is a weak $d(f(x), f(y)) < d(x,y)\ \forall x \neq y$, In fact, we can express (4) in the following manner for any two fixed $x, y \in X$:

$$d(f(x), f(y)) \leq (1-\epsilon)d(x,y) + K\epsilon\varphi(\epsilon), K > 0.$$

Put $\epsilon = 0$, we find

$$d(f(x), f(y)) \leq d(x,y)$$

If equality is achieved, the relationship

$$d(x,y) \leq K\varphi(\epsilon)$$

holds true for all $\epsilon \in [0,1]$, which necessitates that $d(x,y) = 0$.

In order to demonstrate the existence of $x_\star$, we introduce the sequences commencing from x_0.

$$x_j = f\left(x_{j-1}\right) = f^j(x_0) \text{ and } c_j = \|x_j\|.$$

Lemma 3.5. c_j is a bounded sequence.

Proof. take advantage of inequalities

$$\mathrm{d}\left(x_{j+1}, x_j\right) \leq \mathrm{d}\left(x_j, x_{j-1}\right) \leq \mathrm{d}\left(x_{j-1}, x_{j-2}\right) \leq \cdots \leq \mathrm{d}(x_1, x_0) = c_1,$$

The bound is found.

$$c_j \leq \mathrm{d}\left(x_{j+1}, x_1\right) + 2c_1 = \mathrm{d}\left(f\left(x_j\right), f(x_0)\right) + 2c_1.$$

Consequently, as $\beta \leq \alpha$, we infer from (4) that

$$c_j \leq (1-\epsilon)c_j + \lambda\epsilon^\alpha\varphi(\epsilon)\left[1 + c_j + c_1\right]^\beta + 2c_1 \leq (1-\epsilon)c_j + a\epsilon^\alpha\varphi(\epsilon)c_j^\alpha + b$$

for some $a, b > 0$. Accordingly,

$$\epsilon c_j \le a\epsilon^{\alpha}\varphi(\epsilon)c_j^{\alpha} + b$$

The choice $\epsilon = \epsilon_\iota = (1+b)/c_{j_\iota}$ results in a contradiction if there is a subsequence $c_{j_\iota} \to \infty$.

$$1 \le a(1+b)^{\alpha}\varphi(\epsilon_\iota) \to 0$$

Lemma 3.6. The following inequality holds for all $j, i \in \mathbb{N}$.

$$\mathrm{d}\big(x_{j+i}, x_j\big) \le C\omega_j(\alpha)$$

where $c = \sup_{j\in\mathbb{N}} \lambda\big(1+2c_j\big)^{\beta} < \infty$ Due to Lemma 3.5,

Proof. Assume that i is fixed. Given (4), the sequence $p_j = j^{\alpha}\mathrm{d}\big(x_{j+i}, x_j\big)$ satisfies

$$\begin{aligned} p_{j+1} &= (j+1)^{\alpha}\mathrm{d}\Big(f\big(x_{j+i}\big), f\big(x_j\big)\Big) \\ &\le (j+1)^{\alpha}(1-\epsilon)\mathrm{d}\big(x_{j+i}, x_j\big) + c(j+1)^{\alpha}\epsilon^{\alpha}\varphi(\epsilon). \end{aligned}$$

selecting at each j

$$\epsilon = 1 - \left(\frac{j}{j+1}\right)^{\alpha} \le \frac{\alpha}{j+1}$$

we end up with

$$p_{j+1} \le p_j + c\alpha^{\alpha}\varphi\left(\frac{\alpha}{j+1}\right)$$

Since $p_0 = 0$, this gives

$$p_j \le c\alpha^{\alpha}\sum_{\kappa=1}^{j}\varphi\left(\frac{\alpha}{\kappa}\right) \qquad (7)$$

Furthermore, a final division by j^{α} will suffice.

Theorem 3.1 is concluded by Lemma 3.6, which demonstrates that the sequence x_j is a Cauchy with a limit of $x_\star \in X$. Then, through an additional by applying Lemma 3.6,

$$d\left(x_\star, f^{j}(x_0)\right) = d\left(x_\star, f\big(x_{j-1}\big)\right) = d\big(x_\star, x_j\big) = \lim_{i\to\infty} d\big(x_{j+i}, x_j\big) \le C\omega_j(\alpha)$$

As j approaches closer to infinity, we can determine the convergence rate estimate (5) and the equality $x_\star = f(x_\star)$ thanks to the continuity of f. . It remains to demonstrate that c satisfies the necessary constraint. In light of the fact that $x_\star$ is a fixed point, we compose the following:

$$d\big(x_\star, x_j\big) = d\left(f(x_\star), f\big(x_{j-1}\big)\right) \le d\big(x_\star, x_{j-1}\big) \le \cdots \le d(x_\star, x_0) = \|x_\star\|,$$

which entails

$$c_j \leq \|x_\star\| + d(x_\star, x_j) \leq 2\|x_\star\|.$$

We can now verify Corollary 3.2. It is evident that (4) remains valid when $\Theta - \nu$ and ν are substituted for α and γ, respectively, with $\nu > 0$ arbitrarily small for Θ=α+γ. Then,

$$\omega_j(\Theta - \nu) = \frac{(\Theta - \nu)^\Theta}{j^{\Theta-\nu}} \sum_{\kappa=1}^{j} \frac{1}{\kappa^\nu} \leq \frac{(\Theta - \nu)^\Theta}{(1-\nu)j^{\Theta-1}}$$

Then (6) is implied by Theorem 3.1 and an ultimate limit $\nu \to 0$.

4. Fixed Point in Cone Metric Space

In 1905, Fréchet introduced the concept of a metric space [23]. In 1934, Kurepa [37] may have introduced the oldest extension of a metric by utilizing an ordered normed space as the codomain of a metric, as opposed to the set of real numbers. Subsequently, these spaces were utilized by a multitude of mathematicians under a variety of names, like pseudo-metric spaces, extended metric spaces, vector-valued metric spaces, and abstract metric spaces (for more see [43,57]).

The partial ordering is constructed in reference to $\mathcal{K}$, which is a cone in a real Banach space E.

Definition 4.1. Suppose that X is a set that is not empty. This assumes that a mapping $\mathcal{D}: X \times X \to E$ meets the following requirements:

(i) $0 \leq \mathcal{D}(x,y), \forall x, y \in X$ and $\mathcal{D}(x,y) = 0 \Leftrightarrow x = y$.
(ii) $\mathcal{D}(x,y) = \mathcal{D}(y,x)$.
(iii) $\mathcal{D}(x,y) \leq \mathcal{D}(x,z) + \mathcal{D}(z,y), \forall x, y, z \in X$.

The pair $(X, \mathcal{D})$ is referred to as a cone metric space, we refer to it below as CMS., and $\mathcal{D}$ is referred to as a cone metric on X.

-When $d(x,y) \in E$ and E is a Banach space ordered by a cone P, we have a cone metric space $(X, \mathcal{D})$. Open balls defined by the cone's interior create the topology.
-If the cone is normal, the space is always T_0 and Hausdorff.
-Its topology can be reduced to a standard metric topology, making it metrisable.

We direct the reader to references [22], [27], [31], and [43] for additional information on these topological characteristics.

Definition 4.2. A mapping $\mathcal{T}$ when its contraction on $(X, \mathcal{D})$ was a function $\mathcal{T}$ that maps X to itself that satisfies the following condition: $\forall x, y$ in X, $\mathcal{D}(\mathcal{T}(x), \mathcal{T}(y)) \leq c\mathcal{D}(x,y)$.

Sometimes, contraction maps are referred to as Lipschitzian maps, and the Lipschitz constant of $\mathcal{T}$ is the smallest such value of c.

Remark 4.3. The mapping is considered non-expansive if the contraction requirement is satisfied $\forall\, c \leq 1$.

Theorem 4.4. Assume that $\mathcal{K}$ be a normal cone with a normal constant c, and let $(X, \mathcal{D})$ is a complete CMS [27]. We will assume that the contraction condition is satisfied by the mapping $\mathcal{T}: X \to X$:

$\mathcal{D}(\mathcal{T}(x),\mathcal{T}(y)) \leq c\mathcal{D}(x,y), \forall x,y \in X$, at which c is a constant value in the range of 0 to 1. Subsequently, an iterative sequence $\{\mathcal{T}^j(x)\}_{j\geq 1}$ approach to the fixed point for each $x \in X$, and $\mathcal{T}$ has a fixed point in X that is a unique.

Proof: Choosing $x_0 \in X$ and $x_1 = \mathcal{T}(x_o), x_2 = \mathcal{T}(x_1) = \mathcal{T}^2(x_o), x_3 = \mathcal{T}(x_2) = \mathcal{T}^3(x_o), \dots, x_{j+1} = \mathcal{T}(x_j) = \mathcal{T}^{j+1}(x_o)$

We have:

$$\mathcal{D}(x_{j+1}, x_j) = \mathcal{D}(\mathcal{T}x_j, \mathcal{T}x_{j-1}) \leq c\mathcal{D}(x_j, x_{j-1}) \leq c^2\mathcal{D}(x_{j-1}, x_{j-2}) \leq \cdots \leq c^j\mathcal{D}(x_1, x_0)$$

So for $j > i$;

$$\begin{aligned}\mathcal{D}(x_j, x_i) &\leq \mathcal{D}(x_j, x_{j-1}) + \mathcal{D}(x_{j-1}, x_{j-2}) + \cdots + \mathcal{D}(x_{i+1}, x_i) \\ &\leq (c^{j-1} + c^{j-2} + \cdots . c^i)\mathcal{D}(x_1, x_0) \leq \frac{c^i}{1-c}\mathcal{D}(x_1, x_0) \qquad (8)\end{aligned}$$

We can conclude that $\{x_j\}$ is a Cauchy sequence by the normality assumption of $\mathcal{K}$, which states that $\|\mathcal{D}(x_j, x_i)\| \leq \frac{c^i}{1-c}k\|\mathcal{D}(x_1, x_o)\|$. Consequently, $\mathcal{D}(x_j, x_i) \to 0$ (as $j, i \to \infty$). Therefore, the Cauchy convergence property of X means that there is an $x^* \in X$ such that x_j convergence occurs to x^* as j approaches infinity.

Right now ; $\mathcal{D}(\mathcal{T}x^*, x^*) \leq \mathcal{D}(\mathcal{T}x_j, \mathcal{T}x*) + \mathcal{D}(\mathcal{T}x_j, x^*) \leq c\mathcal{D}(x_j, x^*) + \mathcal{D}(x_{j+1}, x^*)$

So; $\|\mathcal{D}(\mathcal{T}x^*, x^*)\| \leq c(c\|\mathcal{D}(x_j, x^*)\| + \|\mathcal{D}(x_{j+1}, x^*)\|) \to 0$

Therefore ; $\|\mathcal{D}(\mathcal{T}x^*, x^*)\| = 0$ consequently $\mathcal{D}(\mathcal{T}x^*, x^*) = 0$

Hence, $\mathcal{T}x^* = x^*$, so x^* which is a fixed point for $\mathcal{T}$.

At this point, we can see that T is unique by allowing y^* to be an additional fixed point.

$\mathcal{D}(x^*, y^*) = \mathcal{D}(\mathcal{T}x^*, \mathcal{T}y^*) \leq c\mathcal{D}(x^*, y^*)$.
Hence, $\|\mathcal{D}(x^*, y^*)\| = 0$ and so $x^* = y^*$.

Theorem 4.5. Consider a sequentially compact CMS $(X, \mathcal{D})$ [27] and a regular cone $\mathcal{K}$. Let $\mathcal{T}: X \to X$ satisfy the condition of contraction: $\mathcal{D}(\mathcal{T}x, \mathcal{T}y) < \mathcal{D}(x, y)$, $\forall\, x, y \in X, x \neq y$. A unique fixed point within X is subsequently encountered by $\mathcal{T}$.

Proof: let $x_\circ \in X$ and $x_1 = \mathcal{T}(x_\circ)$

$$\begin{aligned}x_2 &= \mathcal{T}(x_1) = \mathcal{T}^2(x_0) \\ x_3 &= \mathcal{T}(x_2) = \mathcal{T}^3(x_\circ), \dots., x_{j+1} = \mathcal{T}(x_j) = \mathcal{T}^{j+1}(x_\circ)\end{aligned}$$

If $x_{j+1} = x_j$ for a given j, then x_j is a fixed point on $\mathcal{T}$, and the proof is considered complete.

Therefore, we presuppose that $\forall j$, $x_{j+1} \neq x_j$, and we define d_j as $d(x_j, x_{j+1})$. Subsequently, as follows: $d_{j+1} = \mathcal{D}(x_{j+1}, x_{j+2}) = \mathcal{D}(\mathcal{T}x_j, \mathcal{T}_{x+1}) < \mathcal{D}(x_j, x_{j+1}) = d_j$ Consequently, d_j is a decreasing sequence that is constrained by 0 below. Given that $\mathcal{K}$ is regular, there exists $d^* \in E$ such that $d_j \to d^*$ as n approaches infinity. Therefore, the sequential compactness of X implies that exist subsequences $\{X_{ji}\}$ of $\{x_j\}$ and $x^* \in X$ In this way, $X_{ji} \to X^*$ as i approaches infinity.

We have $\mathcal{D}(\mathcal{T}x_{ji}, \mathcal{T}x^*) \leq \mathcal{D}(x_{ji}, x^*), i = 1,2, \ldots$
So: $\|\mathcal{D}(\mathcal{T}x_{ji}, \mathcal{T}x^*)\| \leq k\|\mathcal{D}(\mathcal{T}x_{ji}, x^*)\| \to 0$(as i tend to ∞)
when the constant c is the normal constant of $\mathcal{K}$.
Therefore ; $\mathcal{T}x_{ji} \to \mathcal{T}x^*, (i \to \infty)$
In the same way, $\mathcal{T}^2 X_{ji} \to \mathcal{T}^2 x^*, (i \to \infty)$
So, $\mathcal{D}(\mathcal{T}x_{ji}, x_{ji}) \to \mathcal{D}(\mathcal{T}x^*, x^*), (i \to \infty)$, as well as, $\mathcal{D}(\mathcal{T}^2 x_{ji}, \mathcal{T}x_{ji}) \to \mathcal{D}(\mathcal{T}^2 x^*, \mathcal{T}x^*), (i \to \infty)$
It is clear that : $\mathcal{D}(\mathcal{T}x_{ji}, x_{ji}) = d_{ji} \to d^* = \mathcal{D}(\mathcal{T}x^*, x^*), (i \to \infty)$
Now, we'll show $\mathcal{T}x^* = x^*$. If $\mathcal{T}x^* \neq x^*$, then $d^* \neq 0$
We have: $d^* = \mathcal{D}(\mathcal{T}x^*, x^*) > \mathcal{D}(\mathcal{T}^2 x^*, \mathcal{T}x^*) = \lim_{i\to\infty} \mathcal{D}(\mathcal{T}^2 x_{ji}, \mathcal{T}x_{ji}) = \lim_{i\to\infty} d_{j+1} = d^*$

There is a contradiction here, which implies that $\mathcal{T}x^* = x^*$. In other words, x^* is a fixed point of a mapping $\mathcal{T}$. The fixed point's distinctiveness is readily apparent.

The normality assumption of the cone $\mathcal{K}$ is omitted in the subsequent theorem, which enhances (Theorem 4.6).

Theorem 4.6. The contraction condition is satisfied by the mapping $\mathcal{T}: X \to X$, where $x, y \in X$, $\mathcal{D}(\mathcal{T}x, \mathcal{T}y) \leq c\mathcal{D}(x, y)$for all $x, y \in X$, where c is a constant $c \in [0,1)$. Assume that $(X, \mathcal{D})$ is a complete CMS [46]. Next, the repeated sequence $\{\mathcal{T}^j x\}_{j\geq 1}$ convergence occurs to the unique fixed point in X , $\forall x \in X$.

Proof: Each as well as every $x_\circ \in X$ to gether with $j \geq 1$
Put $x_1 = \mathcal{T}x_\circ$ as well as $x_{j+1} = \mathcal{T}^{j+1}x_\circ$, subsequently:
$\mathcal{D}(x_{j+1}, x_j) = \mathcal{D}(\mathcal{T}x_j, \mathcal{T}x_{j-1}) \leq c\mathcal{D}(x_j, x_{j-1})$

$$\leq c^2\mathcal{D}(x_{j-1}, x_{j-2}) \leq \cdots \leq c^j\mathcal{D}(x_1, x_\circ) \qquad (9)$$

So that for $j > i$,

$$\mathcal{D}(x_j, x_i) \leq \mathcal{D}(x_j, x_{j-1}) + \mathcal{D}(x_{j-1}, x_{j-2}) + \mathcal{D}(x_{j-2}, x_{j-3}) + \cdots + \mathcal{D}(x_{i+1}, x_i)$$
$$\leq \left(c^{j-1} + c^{j-2} + \cdots + c^i\right)\mathcal{D}(x_1, x_\circ) \leq \frac{c^i}{1-c}\mathcal{D}(x_1, x_0)$$

Now, consider $c \gg 0$ is a given, select $\delta > 0$ such that $c + N_\delta(0) \subseteq \mathcal{K}$, where $N_\delta(0) = \{y \in E: \|y\| < \delta\}$.

Additionally, select a natural number N_1 so that

$\frac{c^i}{1-c}\mathcal{D}(x_1, x_0) \in N_\delta(0), \forall i \geq N_1$. Then; $\frac{c^i}{1-c}\mathcal{D}(x_1, x_0) \ll c, \forall i \geq N_1$
Thus, $\mathcal{D}(x_j, x_i) \leq \frac{c^i}{1-c}\mathcal{D}(x_1, x_0) \ll c, \forall j > i$.

Consequently, ($X, \mathcal{D}$) is a complete CMS, and $\{x_j\}_{j\geq 1}$ is a Cauchy sequence. Consequently, there is $x^* \in X$ such that $x_j \to x^*$.

Select a natural number $N_2 \in \mathbb{N}$, $\mathbb{N}$ is natural number set, such that $\mathcal{D}(x_j, x^*) \ll \frac{c}{2}$ for all j where $j \geq N_2$ Hence, $\mathcal{D}(\mathcal{T}x^*, x^*) \leq \mathcal{D}(\mathcal{T}x_j, \mathcal{T}*) + \mathcal{D}(\mathcal{T}x_j, x^*) \leq c\mathcal{D}(x_j, x^*) + \mathcal{D}(x_{j+1}, x^*) \leq \mathcal{D}(x_j, x^*) + \mathcal{D}(x_{j+1}, x^*) \ll \frac{c}{2} + \frac{c}{2} = c, \forall j \geq N_2$.Thus, $\mathcal{D}(\mathcal{T}x^*, x^*) \ll \frac{c}{i}$, $\forall i \geq 1$. *So* $\frac{c}{i} - \mathcal{D}(\mathcal{T}x^*, x^*) \in \mathcal{K}, \forall i \geq 1$. Since $\frac{c}{i} \to 0$(as $i \to \infty$) and $\mathcal{K}$ is closed, so $-\mathcal{D}(\mathcal{T}x^*, x^*) \in \mathcal{K}$. However, $\mathcal{D}(\mathcal{T}x^*, x^*) \in \mathcal{K}$, therefore $\mathcal{D}(\mathcal{T}x^*, x^*) = 0$ Consequently, $\mathcal{T}x^* = x^*$.

Theorem 4.7. Assume that $(X, \mathcal{D})$ be a CMS that is complete [25]. Set : $B(x_\circ, r) = \{x \in X : \mathcal{D}(x_\circ, x) \leq c\}$for $r \in E$ with $r > 0$ and $x_\circ \in X$. Assume that the mapping $\mathcal{T} : X \to X$ holds for the condition of contraction $\mathcal{D}(\mathcal{T}x, \mathcal{T}y) \leq c\mathcal{D}(x, y)$ for all $x, y \in B(x_\circ, r)$, wherever c is a constant and $0 \leq c < 1$. Then $\mathcal{T}$ has a fixed point in $B(x_\circ, c)$ that is unique if there is a $y \in B(x_\circ, r)$such that: $\mathcal{D}(\mathcal{T}y, y) \leq (1 - c)\big(r - \mathcal{D}(x_\circ, y)\big)$

The succeeding theorem improves The modified normality condition on the cone $\mathcal{K}$ is examined, and Theorem 4.5 is derived by disregarding the regularity assumption.

Theorem 4.8. Suppose that $(X, \mathcal{D})$ be a sequentially compact CMS [25] and $\mathcal{K}$ be a normal cone. And $\mathcal{T} : X \to X$ holds for condition of contraction that : $\mathcal{D}(\mathcal{T}x, \mathcal{T}y) < \mathcal{D}(x, y)$, $\forall x, y \in X$ and $x \neq y$. following that, $\mathcal{T}$ has a unique fixed point in X.

5. **Fixed point in complex valued metric space**

Azam, Fisher, and Khan recently proposed the idea of a complex-valued metric space we refer to it hereinafter as CVMS. that is an extensions of the standard metric space (see [9]). To explicate this, let us consider the following natural connection $\precsim$ on $\mathbb{C}$, the set of complex numbers: for $z_1, z_2 \in \mathbb{C}$

$$z_1 \precsim z_2 \overset{\text{def}}{\Leftrightarrow} \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2) \text{ and } \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2),$$
$$z_1 \prec z_2 \overset{\text{def}}{\Leftrightarrow} \mathrm{Re}(z_1) < \mathrm{Re}(z_2) \text{ and } \mathrm{Im}(z_1) < \mathrm{Im}(z_2).$$

Definition 5.1. suppose that X be a set that is not empty. The following criteria must be met for a mapping $d : X \times X \to \mathbb{C}$ to be referred to be a complex valued metric (CVMS) on X:

(CM1) $0 \precsim d(x, y)$, $\forall x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(CM2) $d(x, y) = d(y, x)$, $\forall x, y \in X$;
(CM3) $d(x, y) \precsim d(x, z) + d(z, y)$, $\forall x, y, z \in X$.

We refer to (X, d) as a complex valued metric space in this particular case.

The fact that this idea is an extensions of the standard metric space is evident. In actuality, d is a metric in the classical sense if $d : X \times X \to \mathbb{R}$ fulfills (CM1)-(CM3), meaning that the subsequent requirements are met:

(M1) $0 \leq d(x, y)$, $\forall\, x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(M3) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall\, x, y, z \in X$.

A number of ideas in the classical theory of metric spaces are analogous to the formulation that follows,

and they are covered in [9]. Other intriguing generalizations of metric spaces exist as well; see [16,17] for one example.

Definition 5.2. Set (X, d) is a CVMS.

- If $\forall\ 0 \prec c \in \mathbb{C}, \exists N$ In this way $d(x_j, x_i) \prec c, \forall j, i \geq N$, N is integer, then the sequence $\{x_j\}$ is a Cauchy.
- If $\forall 0 \prec c \in \mathbb{C}, \exists N$ integer, In this way $d(x_j, x) \prec c$ for all $j \geq N$, then $\{x_j\}$ converges to an element $x \in X$. We write $x_j \xrightarrow{d} x$ in this instance.
- If each Cauchy sequence in X converges to a point in X, we say that (X, d) is complete.

In general :

-The complex absolute value $|\cdot|_c$ is used to define open balls.

-Because sequences can converge to more than one point, the space is typically not Hausdorff.

-Complex distance differences are used to define convergence and closure.

We direct the reader to references [9], [40], and [49] for additional information on these topological characteristics.

Theorem 5.3. Suppose that $0 \leq \lambda, \mu \in \mathbb{R}$, In this way$\lambda + \mu < 1$ and let (X, d) be a full complex valued metric space [9]. Assume that the mappings $F, \mathcal{T}: X \to X$ satisfy:

$$d(Fx, \mathcal{T}y) \precsim \lambda d(x, y) + \frac{\mu d(x, Fx) d(y, \mathcal{T}y)}{1 + d(x, y)}\ \forall x, y \in X. \#(10)$$

A unique common fixed point is possessed by F and $\mathcal{T}$.

In this section, we continue our investigation of theorems of fixed point in CVMS. The results that were obtained are generalizations of recent findings that have been demonstrated in[56] by Sintunavarat and Kumam and in [49] Rouzkard and Imdad. Additionally, we enhance numerous assumptions regarding the mappings that are involved. It is important to mention that [12] has recently proven several distinct fixed point theorems.

Suppose that (X, d) be a CVMS and $F, \mathcal{T}: X \to X$.

Proposition 5.4. assume that $x_0 \in X$ and the sequence $\{x_j\}$ is defined as

$$x_{2j+1} = Fx_{2j}, \#$$
$$x_{2j+2} = \mathcal{T}x_{2j+1},\ \ \forall j = 0, 1, 2, \ldots\ldots \#(11)$$

And Assume that a mapping $\lambda: X \times X \to [0,1)$ satisfying

$$\lambda(\mathcal{T}Sx, y) \leq \lambda(x, y) \text{ and } \lambda(x, F\mathcal{T}y) \leq \lambda(x, y),\ \forall\ x, y \in X.$$

Then $\lambda(x_{2j}, y) \leq \lambda(x_0, y)$ and $\lambda(x, x_{2j+1}) \leq \lambda(x, x_1), \forall\ x, y \in X$ and $j = 0, 1, 2, \ldots.$

Proof: Assume that $x, y \in X$ and $j = 0,1,2, \ldots.$ After that, we've got

$$\lambda(x_{2j}, y) = \lambda(\mathcal{T}Fx_{2j-2}, y) \leq \lambda(x_{2j-2}, y) = \lambda(\mathcal{T}Fx_{2j-4}, y) \leq \cdots \leq \lambda(x_0, y).$$

Also, we have

$$\lambda(x, x_{2j+1}) = \lambda(x, F\mathcal{T}x_{2j-1}) \leq \lambda(x, x_{2j-1}) = \lambda(x, F\mathcal{T}x_{2j-3}) \leq \cdots \leq \lambda(x, x_1).$$

Lemma 5.5. Let $\lambda, \mu: X \times X \to [0,1)$ and $x, y \in X$. If F and $\mathcal{T}$ satisfying

$$d(Fx, \mathcal{T}Fx) \precsim \lambda(x, Fx)d(x, Fx) + \mu(x, Fx)\frac{d(x, Fx)d(Fx, \mathcal{T}Fx)}{1 + d(x, Fx)},$$

$$d(F\mathcal{T}y, \mathcal{T}y) \precsim \lambda(\mathcal{T}y, y)d(\mathcal{T}y, y) + \mu(\mathcal{T}y, y)\frac{d(\mathcal{T}y, F\mathcal{T}y)d(y, \mathcal{T}y)}{1 + d(\mathcal{T}y, y)},$$

then

$$|d(Fx, \mathcal{T}Fx)| \leq \lambda(x, Fx)|d(x, Fx)| + \mu(x, Fx)|d(Fx, \mathcal{T}Fx)|,$$
$$|d(F\mathcal{T}y, \mathcal{T}y)| \leq \lambda(\mathcal{T}y, y)|d(\mathcal{T}y, y)| + \mu(\mathcal{T}y, y)|d(\mathcal{T}y, F\mathcal{T}y)|,$$

respectively.

Proof : write

$$\begin{aligned} |d(Fx, \mathcal{T}Fx)| &\leq \left|\lambda(x, Fx)d(x, Fx) + \mu(x, Fx)\frac{d(x, Fx)d(Fx, \mathcal{T}Fx)}{1 + d(x, Fx)}\right| \\ &\leq \lambda(x, Fx)|d(x, Fx)| + \mu(x, Fx)\left|\frac{d(x, Fx)}{1 + d(x, Fx)}\right||d(Fx, \mathcal{T}Fx)| \end{aligned} \quad (12)$$

Similarly

$$\begin{aligned} |d(F\mathcal{T}y, \mathcal{T}y)| &\leq \left|\lambda(\mathcal{T}y, y)d(\mathcal{T}y, y) + \mu(\mathcal{T}y, y)\frac{d(\mathcal{T}y, F\mathcal{T}y)d(y, \mathcal{T}y)}{1 + d(\mathcal{T}y, y)}\right| \\ &\leq \lambda(\mathcal{T}y, y)|d(y, \mathcal{T}y)| + \mu(\mathcal{T}y, y)\left|\frac{d(y, \mathcal{T}y)}{1 + d(y, \mathcal{T}y)}\right||d(\mathcal{T}y, F\mathcal{T}y)| \end{aligned} \quad (13)$$

Lemma 5.6. Define a sequence $\{x_j\}$ in X and $k \in [0,1)$. If $c_j = |d(x_j, x_{j+1})|$ satisfying

$$c_j \leq kc_{j-1}, \ \forall\, j \in \mathbb{N}, \#(14)$$

then the sequence $\{x_j\}$ is Cauchy.

Proof: when $h \in [0,1)$. Consequently.

$$c_j \leq kc_{j-1} \leq k^2c_{j-2} \leq \cdots \leq k^jc_0, \ \forall j \in \mathbb{N}.$$

For $i, j \in \mathbb{N}$ and $i > j$, then

$$\begin{aligned}\left|d(x_j,x_i)\right| &\le c_j + c_{j+1} + \cdots + c_{i-1}\\ &\le k^j\left(1+k+k^2+\cdots+k^{i-j-1}\right)c_0\\ &\le \frac{k^j}{1-k}c_0\end{aligned}$$

Consequently, the sequence $\{x_j\}$ is Cauchy, as $\left|d(x_j,x_i)\right|$ approaches zero as $j \to \infty$,.

Theorem 5.7. Assume that (X,d) be a CVMS and $F,\mathcal{T}: X \to X$. If $\lambda,\mu,\gamma: X \times X \to [0,1]$ are mappings where for any $x,y \in X$:

Assume that (X,d) be a CVMS that is complete, and let $F,\mathcal{T}: X \to X$. If a mappings λ,μ, and $\gamma: X \times X \to [0,1)$ are exist and meeting the subsequent requirement $\forall\, x,y \in X$:

(a) $\lambda(\mathcal{T}Fx,y) \le \lambda(x,y)$ and $\lambda(x,F\mathcal{T}y) \le \lambda(x,y)$,

$$\mu(\mathcal{T}Fx,y) \le \mu(x,y) \text{ and } \mu(x,F\mathcal{T}y) \le \mu(x,y)$$

$$\gamma(\mathcal{T}Fx,y) \le \gamma(x,y) \text{ and } \gamma(x,F\mathcal{T}y) \le \gamma(x,y);$$

(b) $\lambda(x,y)+\mu(x,y)+\gamma(x,y) < 1$;

(c) $$d(Fx,\mathcal{T}y) \precsim \lambda(x,y)d(x,y) + \mu(x,y)\frac{d(x,Fx)d(y,\mathcal{T}y)}{1+d(x,y)} + \gamma(x,y)\frac{d(y,Fx)d(x,\mathcal{T}y)}{1+d(x,y)}. \#(15)$$

Subsequently, F and $\mathcal{T}$ have a common fixed point and it's unique

Proof: assume that $x,y \in X$. From (15), we take into account

$$\begin{aligned}d(Fx,\mathcal{T}Fx) \precsim& \lambda(x,Fx)d(x,Fx) + \mu(x,Fx)\frac{d(x,Fx)d(Fx,\mathcal{T}Fx)}{1+d(x,Fx)}\\ &+\gamma(x,Fx)\frac{d(Fx,Fx)d(x,\mathcal{T}Fx)}{1+d(x,Fx)}\\ =&\lambda(x,Fx)d(x,Fx) + \mu(x,Fx)\frac{d(x,Fx)d(Fx,\mathcal{T}Fx)}{1+d(x,Fx)}.\end{aligned}$$

The following is derived from Lemma 5.5:

$$|d(Fx,\mathcal{T}Fx)| \le \lambda(x,Fx)|d(x,Fx)| + \mu(x,Fx)|d(Fx,\mathcal{T}Fx)|. \#(16)$$

In the same a direction we obtain

$$\begin{aligned}d(F\mathcal{T}y,\mathcal{T}y) \precsim& \lambda(\mathcal{T}y,y)d(\mathcal{T}y,y) + \mu(\mathcal{T}y,y)\frac{d(\mathcal{T}y,F\mathcal{T}y)d(y,\mathcal{T}y)}{1+d(\mathcal{T}y,y)}\\ &+\gamma(\mathcal{T}y,y)\frac{d(y,F\mathcal{T}y)d(\mathcal{T}y,\mathcal{T}y)}{1+d(\mathcal{T}y,y)}\\ =&\lambda(\mathcal{T}y,y)d(\mathcal{T}y,y) + \mu(\mathcal{T}y,y)\frac{d(\mathcal{T}y,F\mathcal{T}y)d(y,\mathcal{T}y)}{1+d(\mathcal{T}y,y)}.\end{aligned}$$

The following is derived from Lemma 5.5:

$$|d(F\mathcal{T}y,\mathcal{T}y)| \leq \lambda(\mathcal{T}y,y)|d(\mathcal{T}y,y)| + \mu(\mathcal{T}y,y)|d(\mathcal{T}y,F\mathcal{T}y)|. \#(17)$$

For the sequence $\{x_j\}$, let $x_0 \in X$ and be defined by (11). We demonstrate that $\{x_j\}$ is a Cauchy sequence. For all $h = 0, 1, 2, \ldots$, according to Proposition 5.4, (16), (17), we have

$$\begin{aligned}
|d(x_{2h+1},x_{2h})| =& |d(F\mathcal{T}x_{2h-1},\mathcal{T}x_{2h-1})| \\
\leq& \lambda(\mathcal{T}x_{2h-1},x_{2h-1})|d(\mathcal{T}x_{2h-1},x_{2h-1})| \\
&+\mu(\mathcal{T}x_{2h-1},x_{2h-1})|d(\mathcal{T}x_{2h-1},F\mathcal{T}x_{2h-1})| \\
=& \lambda(x_{2h},x_{2h-1})|d(x_{2h-1},x_{2h})| + \mu(x_{2h},x_{2h-1})|d(x_{2h},x_{2h+1})| \\
\leq& \lambda(x_0,x_{2h-1})|d(x_{2h-1},x_{2h})| + \mu(x_0,x_{2h-1})|d(x_{2h+1},x_{2h})| \\
\leq& \lambda(x_0,x_1)|d(x_{2h-1},x_{2h})| + \mu(x_0,x_1)|d(x_{2h+1},x_{2h})|,
\end{aligned}$$

Which means

$$|d(x_{2h+1},x_{2h})| \leq \frac{\lambda(x_0,x_1)}{1-\mu(x_0,x_1)}|d(x_{2h-1},x_{2h})| \#(18)$$

In the same a direction we obtain

$$\begin{aligned}
|d(x_{2h+2},x_{2h+1})| &= |d(\mathcal{T}Fx_{2h},Fx_{2h})| \\
&\leq \lambda(x_{2h},Fx_{2h})|d(x_{2h},Fx_{2h})| + \mu(x_{2h},Fx_{2h})|d(Fx_{2k},\mathcal{T}Fx_{2k})| \\
&= \lambda(x_{2h},x_{2h+1})|d(x_{2h},x_{2h+1})| + \mu(x_{2h},x_{2h+1})|d(x_{2h+1},x_{2h+2})| \\
&\leq \lambda(x_0,x_{2h+1})|d(x_{2h},x_{2h+1})| + \mu(x_0,x_{2h+1})|d(x_{2h+2},x_{2h+1})| \\
&\leq \lambda(x_0,x_1)|d(x_{2h},x_{2h+1})| + \mu(x_0,x_1)|d(x_{2h+2},x_{2h+1})|,
\end{aligned}$$

Which means

$$|d(x_{2h+2},x_{2h+1})| \leq \frac{\lambda(x_0,x_1)}{1-\mu(x_0,x_1)}|d(x_{2h},x_{2h+1})| \#(19)$$

Assume that $\mathcal{L} = \frac{\lambda(x_0,x_1)}{1-\mu(x_0,x_1)} < 1$. Then we have

$$|d(x_{j+1},x_j)| \leq \mathcal{L}|d(x_{j-1},x_j)|, \text{ for all } j \in \mathbb{N}. \#(20)$$

We can conclude that sequence $\{x_j\}$ is a Cauchy in (X,d) from Lemma 5.6. By the Cauchy convergence property of X, there is a $z \in X$ where $x_j \to z$ as $j \to \infty$.

Subsequently, we demonstrate that z is a fixed point of F. As a result of Proposition 5.4 and (15), we have

$$\begin{aligned}
d(z,Fz) &\lesssim d(z,\mathcal{T}x_{2j+1}) + d(\mathcal{T}x_{2j+1},Fz) \\
&= d(z,x_{2j+2}) + d(Fz,\mathcal{T}x_{2j+1}) \\
&\lesssim d(z,x_{2j+2}) + \lambda(z,x_{2j+1})d(z,x_{2j+1}) + \mu(z,x_{2j+1})\frac{d(z,Fz)d(x_{2j+1},\mathcal{T}x_{2j+1})}{1+d(z,x_{2j+1})} \\
&+\gamma(z,x_{2j+1})\frac{d(x_{2j+1},Fz)d(z,\mathcal{T}x_{2j+1})}{1+d(z,x_{2j+1})}
\end{aligned}$$

$$\precsim d\left(z,x_{2j+2}\right)+\lambda(z,x_1)d\left(z,x_{2j+1}\right)+\mu(z,x_1)\frac{d(z,Fz)d\left(x_{2j+1},x_{2j+2}\right)}{1+d\left(z,x_{2j+1}\right)}$$
$$+\gamma(z,x_1)\frac{d\left(x_{2j+1},Fz\right)d\left(z,x_{2j+2}\right)}{1+d\left(z,x_{2j+1}\right)}.$$

Consequently, $d(z,Fz)=0$. therefore, $z=Fz$.

Additionally, we demonstrate that z is a fixed point of $\mathcal{T}$. In the case of (15), we have

$$\begin{aligned}d(z,\mathcal{T}z)\precsim& d\left(z,Fx_{2j}\right)+d\left(Fx_{2j},\mathcal{T}z\right)\\ \precsim& d\left(z,x_{2j+1}\right)+\lambda\left(x_{2j},z\right)d\left(x_{2j},z\right)+\mu\left(x_{2j},z\right)\frac{d\left(x_{2j},Fx_{2j}\right)d(z,\mathcal{T}z)}{1+d\left(x_{2j},z\right)}\\ &+\gamma\left(x_{2j},z\right)\frac{d\left(z,Fx_{2j}\right)d\left(x_{2j},\mathcal{T}y\right)}{1+d\left(x_{2j},z\right)}\\ \precsim& d\left(z,x_{2j+1}\right)+\lambda(x_0,z)d\left(x_{2j},z\right)+\mu(x_0,z)\frac{d\left(x_{2j},x_{2j+1}\right)d(z,\mathcal{T}z)}{1+d\left(x_{2j},z\right)}\\ &+\gamma(x_0,z)\frac{d\left(z,x_{2j+1}\right)d\left(x_{2j},\mathcal{T}y\right)}{1+d\left(x_{2j},z\right)}.\end{aligned}$$

Consequently, $d(z,\mathcal{T}z)=0$, indicating that $z=\mathcal{T}z$. Consequently, z is a shared fixed point of F and $\mathcal{T}$. Ultimately, we demonstrate the quality of being distinctive. Assume that there is a $z^*\in X$ such that $z^*=Fz^*=\mathcal{T}z^*$. Then

$$\begin{aligned}d(z,z^*)=&\,d(Fz,\mathcal{T}z^*)\\ \precsim&\,\lambda(z,z^*)d(z,z^*)+\mu(z,z^*)\frac{d(z,Fz)d(z^*,\mathcal{T}z^*)}{1+d(z,z^*)}+\gamma(z,z^*)\frac{d(z^*,Fz)d(z,\mathcal{T}z^*)}{1+d(z,z^*)}\\ =&\,\lambda(z,z^*)d(z,z^*)+\gamma(z,z^*)\frac{d(z^*,z)d(z,z^*)}{1+d(z,z^*)}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}|d(z,z^*)|\le&\,\lambda(z,z^*)|d(z,z^*)|+\gamma(z,z^*)|d(z,z^*)|\left|\frac{d(z,z^*)}{1+d(z,z^*)}\right|\\ \le&\,\lambda(z,z^*)|d(z,z^*)|+\gamma(z,z^*)|d(z,z^*)|\\ \le&\,\left(\lambda(z,z^*)+\gamma(z,z^*)\right)|d(z,z^*)|.\end{aligned}$$

Since $\lambda(z,z^*)+\gamma(z,z^*)<1$, we have $|d(z,z^*)|=0$. Thus $z=z^*$.

The following corollary is derived by setting $F=\mathcal{T}$ in Theorem 5.7.

Corollary 5.8. Assume that (X,d) be CVMS that is complete, and let $\mathcal{T}:X\to X$. If there are mappings λ,μ, and $\gamma:X\times X\to[0,1)$ that satisfy the following condition $\forall x,y\in X$:

(a) $\lambda(\mathcal{T}x,y)\le\lambda(x,y)$ and $\lambda(x,\mathcal{T}y)\le\lambda(x,y)$,
$\mu(\mathcal{T}x,y)\le\mu(x,y)$ and $\mu(x,\mathcal{T}y)\le\mu(x,y)$,
$\gamma(\mathcal{T}x,y)\le\gamma(x,y)$ and $\gamma(x,\mathcal{T}y)\le\gamma(x,y)$;

(b) $\lambda(x,y)+\mu(x,y)+\gamma(x,y)<1$;

(c) $$d(\mathcal{T}x,\mathcal{T}y) \precsim \lambda(x,y)d(x,y)+\mu(x,y)\frac{d(x,\mathcal{T}x)d(y,\mathcal{T}y)}{1+d(x,y)}+\gamma(x,y)\frac{d(y,\mathcal{T}x)d(x,\mathcal{T}y)}{1+d(x,y)}.\#(21)$$

There is a unique common fixed point of $\mathcal{T}$.

The following corollary is derived by selecting $\gamma=0$ in Theorem 5.7.

Corollary 5.9. Set $F,\mathcal{T}: X\to X$ and (X,d) be a complete complex valued metric space. if there are mappings $\lambda,\gamma: X\times X\to[0,1)$ that satisfy the following condition $\forall\, x,y\in X$:

(a) $\lambda(\mathcal{T}Fx,y)\le\lambda(x,y)$ and $\lambda(x,F\mathcal{T}y)\le\lambda(x,y)$, $\gamma(\mathcal{T}Fx,y)\le\gamma(x,y)$ and $\gamma(x,F\mathcal{T}y)\le\gamma(x,y)$;
(b) $\lambda(x,y)+\gamma(x,y)<1$;

(c) $$d(Fx,\mathcal{T}y) \precsim \lambda(x,y)d(x,y)+\gamma(x,y)\frac{d(y,Fx)d(x,\mathcal{T}y)}{1+d(x,y)}.\#(22)$$

Subsequently, F and $\mathcal{T}$ share common fixed point that is unique.

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