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FULL PAPER

Lie Derivative Approach to Scalar Quantities of Tensors in $G\mathcal{B}K-5$ Recurrent Finsler Space

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Abstract

This paper investigates scalar quantities of certain types of tensors in generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald, which admits the curvature inheritance. We established scalar values of the vector field of fifth-order when it moves in the same or in the opposite direction of the coordinates (x^i) by using Lie-derivative within this space. Also, we obtain the value of the curvature scalar K and its Berwald covariant derivative of fifth order when the fundamental function Finsler space $F(x,\dot{x})$ is independent of the positional coordinate and depends only on the directional

coordinate.

Keywords: $G\mathcal{B}K$-5recurrent Finsler space, Lie-derivative L_v , Scalar function of tensor, Curvature inheritance, K −curvature, H −curvature, Finsler Geometry

1. Introduction

Finsler geometry, a natural generalization of Riemannian geometry, has become an essential framework for exploring geometrical structures where the metric depends on both position and direction. The Lie derivative offers a powerful means of examining the behavior of tensor fields along vector fields, making it a suitable approach for identifying inherited properties in higher-order recurrent spaces. We investigate how scalar functions of various curvature tensors transform or remain invariant under Lie differentiation. Specific attention is given to K −curvature, H −curvature, and R −curvature tensors, whose scalar contractions are analytically computed under $G\mathcal{B}K$-5recurrence conditions. Our approach also involves the use of tensor contraction techniques, scalarization procedures, and symmetry arguments supported by key lemmas. A series of works by Al-Qashbari et al. [2–9] have significantly advanced the understanding of generalized $\mathcal{B}K$-fifth recurrent Finsler spaces $G\mathcal{B}K$-5RF_n , particularly through the application of Lie derivatives and Berwald covariant tensors. These contributions have introduced new insights into curvature inheritance properties, projective motions, and curvature tensor decompositions. In particular, the works [4, 5] focused on the inheritance properties of Kulkarni–Nomizu products and extensions of $G\mathcal{B}K$-5RF_n , while [6] analyzed the behavior of conharmonic curvature tensors under Lie differentiation.

Abdallah et al. [1, 12, 14, 16] explored the properties of Finsler spaces, especially in relation to higher-order Berwald connections and decomposition of curvature tensors. These findings have provided foundational mathematical tools used in this paper. Additionally, the structure of Lie-recurrent Finsler manifolds and generalized H −recurrent, as studied by Pandey and collaborators [19–21], laid theoretical groundwork for understanding scalar functions and tensor behavior under recurrent curvature conditions.

Other researchers have focused on the geometric implications of curvature inheritance in various Finsler spaces. For instance, Singh and colleagues [17, 23] investigated $W-$ and $H-$curvature inheritance in bi-recurrent spaces, offering essential criteria for identifying scalar curvature behavior. Gouin [13] and Struckmeier et al. [24] contributed to the theoretical basis for applying Lie derivatives in broader physical and geometrical contexts.

Furthermore, foundational mathematical concepts such as tensor contraction, outer products, and scalar-valued functions of tensors are grounded in classical works like Murray [15] and expanded upon in the context of modern Finsler geometry by Singh et al. [22], as well as Ohta [18]. The current research builds upon these concepts to develop new scalar identities and theorems in $G\mathcal{B}K$-5RF_n using Lie-derivatives and scalar functions of tensors.

In summary, the present work is a natural extension of these earlier efforts, unifying and generalizing several results in the literature to investigate the scalar characteristics of curvature tensors

in fifth-order recurrent Finsler spaces under specific geometric and algebraic conditions. The objective of this paper is find a scalar value for tensors in Finsler space. This scalar quantity helps us understand the directionality of atmospheric disturbance at different points and identify high and low-pressure area.

2. Preliminaries

In this section, we consider a generalized $\mathcal{B}K$-fifth recurrent Finsler space, denoted as $G\mathcal{B}K-5RF_n$, that satisfies the following relation [7]

(2.1) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm}K^i_{jkh} + b_{sqlnm}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) - c_{sqlnm}\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) - d_{sqlnm}\left(\delta^i_h C_{jkl} - \delta^i_k C_{jhl}\right) - e_{sqlnm}\left(\delta^i_h C_{jkq} - \delta^i_k C_{jhq}\right) - 2b_{qlnm}y^r\mathcal{B}_r\left(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\right),$$

where $K^i_{jkh} \neq 0$, a_{sqlnm}, b_{sqlnm}, c_{sqlnm} , d_{sqlnm} and e_{sqlnm} are no-zero vector fields of fifth order.

(2.2) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K = a_{sqlnm}K ,$$

if and only if

(2.3) $$nb_{sqlnm} - \left[c_{sqlnm}C_n + d_{sqlnm}C_l + e_{sqlnm}C_q + 2b_{qlnm}y^r\mathcal{B}_r C_s\right] = 0.$$

Also,

(2.4) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K_{jk} = a_{sqlnm}K_{jk},$$

if and only if

(2.5) $$b_{sqlnm}g_{jk} - c_{sqlnm}C_{jkn} - d_{sqlnm}C_{jkl} - e_{sqlnm}C_{jkq} - 2b_{qlnm}y^r\mathcal{B}_r C_{jks} = 0.$$

Also,

(2.6) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H = a_{sqlnm}H + b_{sqlnm}F^2.$$

And

(2.7) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L_v K^i_{jkh}\right) = L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh}\right),$$

if and only if

(2.8) $$L_v a_{sqlnm} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\alpha(x) = 0 \quad .$$

The metric tensor g_{ij} and Kronecker delta δ^i_h are satisfying the relation:

(2.9) $$g_{ij}\,g^{ik} = \delta^k_j = \begin{cases} 1 & if\ j = k \\ 0 & if\ j \neq k \end{cases} .$$

The Cartan's fourth curvature tensor K^i_{jkh}, Cartan's third curvature tensor R^i_{jkh}, $h(v)-$torsion tensor H^i_{kh}, deviation tensor H^i_h, K -Ricci tensor K_{jk} and curvature vector H_k satisfy the following relations

(2.10) a) $K^i_{jkh}y^j = R^i_{jkh}y^j = H^i_{kh}$ b) $K^i_{jkh} = R^i_{jkh} - C^i_{js}H^s_{kh}$ c) $H^r_r = (n-1)H$

d) $H^i_{kh}y^k = H^i_h$ e) $K^r_{jkr} = K_{jk}$ f) $K_{jk}y^j = H_k$ g) $K_{jk} = R_{jk} - C^r_{js}H^s_{kr}$

h) $R_{jk}g^{jk} =$ i) $K_{jk}g^{jk} = K$ j) $R^i_{jkh}\, g^{jk} = R^i_h$ k) $H^i_{jkh}y^j = H^i_{kh}$.

The fundamental function of Finsler space is given by

(2.11) $F^2 = y^i y_i$,

where the vectors y^i and y_i are non-zero. The vector y^k is given by

(2.12) $y^k = g^{jk}y_j$

A Lie- derivative evaluate the rate of change of avector field or a tensor field along the smooth vector field $v^i(x)$. The Lie-derivative of a general mixed tensor field T^i_{jkh} is given by

(2.13) $$L_v T^i_{jkh} = v^m \mathcal{B}_m T^i_{jkh} - T^m_{jkh} \mathcal{B}_m v^i + T^i_{mkh} \mathcal{B}_j v^m + T^i_{jmh} \mathcal{B}_k v^m + T^i_{jkm} \mathcal{B}_h v^m + \dot{\partial}_m T^i_{jkh} \mathcal{B}_r v^m y^r \quad ,$$

where $v^m \neq 0$ is a contravariant vector field independent of directional argument and dependent on positional coordinates (x^i) only and $\mathcal{B}_j v^m = 0$.

The Lie-derivative of the metric tensor g_{ij} is vanishing , i.e.

(2.14) a) $L_v g^{jk} = 0$ and b) $L_v y^j = 0$.

The H-curvature inheritance, K-curvature inheritance and R-curvature inheritance are given by

(2.15) a) $L_v H^i_{jkh} = \alpha(x) H^i_{jkh}$ b) $L_v K^i_{jkh} = \alpha(x) K^i_{jkh}$ and c) $L_v R^i_{jkh} = \alpha(x) R^i_{jkh}$

The scalar function of tensor $\alpha(x)T$ it's a function whose inputs are n-dimensional tensors, but its output is scalar tensor. The scalar function $\alpha(x)$ of tensor $T(x, \dot{x})$ of different order is given by [17, 24]

(2.16) a) $\alpha(x)T_k = \sum_{k=1}^n T_k y^k$

b) $\alpha(x)T^k = \sum_{k=1}^n T^k y_k$

c) $\alpha(x)T^i_j = \sum_{i=1}^n T^i_i$

d) $\alpha(x)T_{jk} = \sum_{j,k=1}^n T_{jk} g^{jk}$

e) $\alpha(x)T^{jk} = \sum_{j,k=1}^n T^{jk} g_{jk}$

f) $\alpha(x)T^i_{kh} = \alpha(x)T^i_{ki} = \alpha(x)T_k = \sum_{k=1}^n T_k y^k$

g) $\alpha(x)T^i_{jkh} = \alpha(x)T^i_{jki} = \alpha(x)T_{jk} = \sum_{j,k=1}^n T_{jk} g^{jk}$

h) $\alpha(x)T^{ir}_{jkh} = \alpha(x)T^{ir}_{jkr} = \alpha(x)T^i_{jk} = \alpha(x)T^i_{ji} = \alpha(x)T_j = \sum_{j=1}^n T_j y^j$.

The curvature scalar H satisfies the following relations

(2.17) a) $L_v H = \alpha(x) H$ and b) $\mathcal{B}_m H = \lambda_m H$.

A Finsler space whose Berwald connection parameter G^i_{kh} is independent of y^i is called an affinely connected space (Berwald space), is characterized by the condition [10, 11]

(2.18) $\mathcal{B}_k g^{ij} = 0$.

Lemma 2.1. The scalar functions of two tensors of the same type is equal [15]

Lemma 2.2: The outer product of two tensors of type (r_1 , s_1) and (r_2 , s_2) is a tensor of type $(r_1 + r_2 , s_1 + s_2)$ [15] .

Lemma 2.3. If in a tensor of order q,where n-upper components are equal n-lower components, then the result is a tensor of order q-$2n$, this process is called (contraction) [15].

Lemma 2.4. A flat Finsler spacetime is a Finsler space where the fundamental function of Finsler space $F(x, \dot{x})$ is independent of the positional coordinate and depends only on the directional coordinate, there is no curvature, and consequently the derivative of curvature tensor is zero [25].

3. Main Results

In this section, we obtained a scalar values for tensors in generalized fifth recurrent Finsler space by using Lie-derivative and scalar function of tensors. Using (2.9) in (2.1), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K^i_{jkh} = a_{sqlnm}K^i_{jkh} .$$

Taking the Lie- derivative of both sides of above equation, then using (2.7) and (2.8) in the result equation, we get

(3.1) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L_v K^i_{jkh}\right) = a_{sqlnm}L_v K^i_{jkh} .$$

Let as assume

(3.2) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L_v K^i_{jkh}\right) = (L_v K^i_{jkh})^2.$$

Using (3.2) in (3.1), we get

(3.3) $$L_v K^i_{jkh} = a_{sqlnm}.$$

Using [(2.15)b] in above equation, we get

$$a_{sqlnm} = \alpha(x)K^i_{jkh} .$$

Using [(2.16)g] in above equation, we get

(3.4) $$a_{sqlnm} = \sum_{j,k=1}^{n} K_{jk}g^{jk}.$$

Theorem 3.1. *In GBK-$5RF_n$, which admits the K-curvature inheritance, the vector field of fifth-order a_{sqlnm} moves a distance of $\sum_{j,k=1}^{n} K_{jk}g^{jk}$ in the same direction of the coordinates (x^i) [provided(3.2) holds].*

From [(2.10)a], we get

$$K^i_{jkh}y^j = R^i_{jkh}y^j.$$

Using (2.12) in above equation, and since $y_k \neq 0$, we get

$$K^i_{jkh}g^{jk} = R^i_{jkh}g^{jk}.$$

Using [(2.10)j] in above equation, we get

(3.5) $$K^i_{jkh}g^{jk} = R^i_h.$$

Transvecting (3.3) by metric tensor g^{jk} , using [(2.14)a] and (3.5), we get

(3.6) $L_v R_h^i = a_{sqlnm}\, g^{jk}$.

Transvecting [(2.15)c] by metric tensor g^{jk} ,using [(2.14)a] and [(2.10)j], we get

$$L_v R_h^i = \alpha(x) R_h^i.$$

Using above equation in (3.6), we get

$$\alpha(x) R_h^i = a_{sqlnm}\, g^{jk}.$$

Using [(2.16)c] and (3.4) in above equation, we get

Theorem 3.2. *In $GBK\text{-}5RF_n$,which admits the K-curvature inheritance and R-curvature inheritance, the scalar value of the metric tensor* $g^{jk} = \frac{\sum_{i=1}^{n} R_i^i}{\sum_{j,k=1}^{n} K_{jk} g^{jk}}$ *[provided(3.2) holds].*

Transvecting (3.3) by the vector y^j , using [(2.14)b] and [(2.10)a], we get

(3.7) $L_v H_{kh}^i = a_{sqlnm} y^j$.

Transvecting [(2.15)a] by the vector y^j , using [(2.14)b] and [(2.10)k], we get

$$L_v H_{kh}^i = \alpha(x) H_{kh}^i$$

Using above equation in (3.7), we get

(3.8) $\alpha(x) H_{kh}^i = a_{sqlnm} y^j$.

Using [(2.16)f] and (3.4) in above equation, we get

Theorem 3.3. *In $GBK\text{-}5RF_n$, which admits the K-curvature inheritance and H-curvature inheritance, the scalar value of the vector* $y^j = \frac{\sum_{k=1}^{n} H_k y^k}{\sum_{j,k=1}^{n} K_{jk} g^{jk}}$ *[provided(3.2) holds].*

Taking the scalar function of both sides of [(2.10)g], we get

$$\alpha(x) C_{js}^r H_{kr}^s = \alpha(x) R_{jk} - \alpha(x) K_{jk}.$$

By using [lemma 2.2] and [lemma 2.3] in above equation, we get

$$\alpha(x) A_{jk} = \alpha(x) R_{jk} - \alpha(x) K_{jk},$$

where $A_{jk} = A_{jskr}^{rs} = C_{js}^r H_{kr}^s$.

Using [(2.16)d] in the right side of above equation, we get

$$\alpha(x) A_{jk} = \sum_{j,k=1}^{n} (R_{jk} - K_{jk}) g^{jk}.$$

Transvecting above equation by y^k, we get

$$\alpha(x) A_{jk} y^k = y^k \sum_{j,k=1}^{n} (R_{jk} - K_{jk}) g^{jk}.$$

By using [lemma 2.2] and [lemma 2.3] in above equation, we get

$$\alpha(x) B_j = y^k \sum_{j,k=1}^{n} (R_{jk} - K_{jk}) g^{jk} ,$$

where $B_j = B_{jk}^k = A_{jk} y^k$.

Transvecting above equation by y_k , using (2.11) and [(2.10)g], we get

$$\alpha(x)B_j y_k = F^2 \sum_{j,k=1}^{n} A_{jk}\ \ g^{jk}\ .$$

Which by [lemma 2.2] can be written as

$$\alpha(x)E_{jk} = F^2 \sum_{j,k=1}^{n} A_{jk}\ \ g^{jk}\ ,$$

where $E_{jk} = B_j y_k$.

Using [(2.16)d] in above equation, we get

Theorem 3.4. *In $GBK\text{-}5RF_n$, which admits the R-curvature inheritance, the scalar value of the fundamental function of Finsler space $F = \sqrt{\frac{\sum_{j,k=1}^{n} E_{jk} g^{jk}}{\sum_{j,k=1}^{n} A_{jk}\ g^{jk}}}$, where $E_{jk} = A_{jk} y^k y_k$ and $A_{jk} = C_{js}^{r} H_{kr}^{s}$.*

Taking the Lie- derivative of both sides of [(2.10)b], we get

$$L_v\, K_{jkh}^{i} = L_v R_{jkh}^{i} - L_v C_{js}^{i} H_{kh}^{s}.$$

Using (3.3) in above equation, we get

$$a_{sqlnm} = L_v R_{jkh}^{i} - L_v (C_{js}^{i} H_{kh}^{s}).$$

Using [(2.15)c] in above equation, we get

$$a_{sqlnm} = \alpha(x) R_{jkh}^{i} - L_v (C_{js}^{i} H_{kh}^{s}).$$

Using [(2.16)g] and (3.4) in above equation, we get

$$L_v (C_{js}^{i} H_{kh}^{s}) = \sum_{j,k=1}^{n} (R_{jk}\ - K_{jk}) g^{jk}.$$

Thus,

Theorem 3.5. *In $GBK\text{-}5RF_n$, which admits the R-curvature inheritance and K-curvature inheritance, the scalar value of the Lie -derivative of tensor $(C_{js}^{i} H_{kh}^{s})$ is equal $\sum_{j,k=1}^{n} (R_{jk} - K_{jk}) g^{jk}$ [provided(3.2) holds].*

From [lemma 2.1], we have

$$\alpha(x) H_h^i = \alpha(x) R_h^i.$$

Transvecting above equation by y_k and if the tensor $(R_h^i y_k)$ is symmetric in the lower indices h and k, we get

$$y_k \alpha(x) H_h^i = \alpha(x) R_k^i y_h.$$

Transvecting above equation by y^k and using (2.11), we get

$$F^2 \alpha(x) H_h^i = \alpha(x) R_k^i y_h y^k.$$

Which by [lemma 2.2] and [lemma 2.3] can be written as

$$F^2 \alpha(x) H_h^i = \alpha(x) Q_h^i,$$

where $Q_h^i = Q_{kh}^{ik} = R_k^i y_h y^k$.

Using [(2.16)c] in above equation, we get

$$F^2 \sum_{i=1}^{n} H_i^i = \sum_{i=1}^{n} Q_i^i.$$

Which can be written as

(3.9) $$F^2 = \frac{\sum_{i=1}^{n} Q_i^i}{\sum_{i=1}^{n} H_i^i}.$$

Using [lemma 2.4] in (2.6), we get

(3.10) $$a_{sqlnm} H + b_{sqlnm} F^2 = 0.$$

Transvecting (3.8) by y^k and using [(2.10)d], we get

$$\alpha(x) H_h^i = a_{sqlnm} y^j y^k.$$

Contracting the indices i and h in above equation and using [(2.10)c], we get

$$(n-1)\alpha(x) H = a_{sqlnm} y^j y^k.$$

Using [(2.17)a] and (2.13) in above equation, we get

$$(n-1) v^m \mathcal{B}_m H = a_{sqlnm} y^j y^k.$$

Using [(2.17)b] in above equation, we get

(3.11) $$H = a_{sqlnm},$$

if

(3.12) $$y^j y^k = (n-1) v^m \lambda_m.$$

Now, using (3.11), (3.4) and (3.9) in (3.10), we get

Theorem 3.6. *In $GBK\text{-}5RF_n$, which admits the R-curvature inheritance and H-curvature inheritance, the vector field of fifth-order b_{sqlnm} moves a distance of* $\frac{(\sum_{j,k=1}^{n} K_{jk} g^{jk})^2 \sum_{i=1}^{n} H_i^i}{\sum_{i=1}^{n} Q_i^i}$, *where $Q_h^i = R_k^i y_h y^k$ in the opposite direction of the coordinates (x^i) if the tensor $(R_h^i y_k)$ is symmetric in the lower indices h and k [provided (3.2) and (3.12) hold].*

From [lemma 2.1], we have

$$\alpha(x) K_{jk} = \alpha(x) R_{jk}.$$

Contracting the indices i and h in [(2.15)b] and using [(2.10)e], then using the result equation in above equation , we get

$$L_v K_{jk} = \alpha(x) R_{jk}.$$

Transvecting above equation by g^{jk} , using [(2.10)h,i] and [(2.14)a], we get

$$L_v K = \alpha(x) R.$$

Using (2.13) in above equation, we get

$$v^m \mathcal{B}_m K = \alpha(x) R.$$

Taking the Berwald covariant derivative of fourth order for above equation with respect to x^n, x^l, x^q and x^s, we get

(3.13) $$v^m \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n \mathcal{B}_m K = \alpha(x) \mathcal{B}_s \mathcal{B}_q \mathcal{B}_l \mathcal{B}_n R.$$

If the curvature scalar R behaves as fourth recurrent, this implies that

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n R = a_{sqln}R.$$

Using above equation in (3.13), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K = \alpha(x)\omega^{sq}_{lnm}R \quad , \quad \text{where } \omega^{sq}_{lnm} = \frac{a_{sqln}}{v^m}.$$

By using [lemma 2.2] in above equation, we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K = \alpha(x)\varphi^{sq}_{lnm},$$

where $\varphi^{sq}_{lnm} = \omega^{sq}_{lnm}R$.

Using [(2.16)h] and (2.2) in above equation, we get

$$a_{sqlnm}K = \sum_{l=1}^{n}\varphi_l y^l$$

Using (3.4) in above equation, we get

(3.14) $$K = \frac{\sum_{l=1}^{n}\varphi_l y^l}{\sum_{j,k=1}^{n}K_{jk}g^{jk}},$$

where $\varphi_l = \frac{Ra_l}{v}$.

Theorem 3.7. *In GBK-$5RF_n$, which admits the K-curvature inheritance, scalar value of the curvature scalar K* that given by (3.14) *if the curvature scalar R behaves as fourth recurrent [provided (3.2) and (2.3) hold].*

Now, we have corollary related to the previous theorem. Transvecting (2.4) by g^{jk}, using [(2.10) i] and (2.18), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K = a_{sqlnm}K.$$

Using (3.14) and (3.4) in above equation, we get

(3.15) $$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m K = \sum_{l=1}^{n}\varphi_l y^l.$$

Thus, we conclude

Corollary 3.1. *In GBK-$5RF_n$, which admits the K-curvature inheritance ,the scalar value of the Berwald covariant derivative of fifth order for the curvature scalar K is given by (3.15), if the curvature scalar R behaves as fourth recurrent [provided (3.2), (2.3) and (2.5) hold].*

4. Conclusions

In this paper we established several scalar values of some tensors and the fundamental function Finsler space $F(x,\dot{x})$ in generalized fifth recurrent Finsler space by using Lie-derivative and scalar function of tensor under certain conditions.

5. Significance of Study

In previous study, we obtained scalar values of some tensors in generalized fifth recurrent Finsler space by using Lie-derivative. The importance of this study is to determine the direction and magnitude of the tensors when a force acts upon it, which helps us understand many physical and engineering phenomena in our lives.

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