ISSN (Print) 2794-7629 Received 10/04/2025
ISSN (Online) 2794- 4549 Accepted 26/06/2025

FULL PAPER

Isomorphic Classifications of Compact Operators' Spaces

Prepared by

Muntasir Suhail
Department of Mathematics
College of Science
Qassim University, Buraydah 51452
Saudi Arabia
m.suhail@qu.edu.sa

Abstract

A few Banach spaces are thoroughly isomorphically classified in this paper. The space of all continuous Y-valued functions defined in the interval of ordinals $[1, \rho + \varepsilon]$ and furnished with the supremum norm is $\mathcal{K}(X, Y^{\rho+\varepsilon})$, we consider compact operators from X to $Y^{\rho+\varepsilon}$, where $\rho + \varepsilon \geq \omega$, Specifically, we extend a recent result of C. Samuel under the Continuum Hypothesis by classifying spaces $\mathcal{K}(X^{\xi}, C_0(\Omega)^{\rho+\varepsilon})$, up to isomorphism, where $\omega \leq \xi < \alpha + \varepsilon$, $\rho + \varepsilon \geq \omega$, Ω is a countable set, X contains no complemented copy of l_1, X^* has the Mazur property, and $X^{**} \leq \aleph_1$.

Keywords: Isomorphic Banach spaces of continuous functions , compact operators.

The Scholar Journal for Sciences & Technology sjst

1. Introduction

Let U_n be a compact Hausdorff space and X be a Banach space. The Banach space of all continuous X -valued functions defined on U_n and furnished with the supremum norm is represented by the notation $C(U_n, X)$. In the scenario where $X = R$. this space will be represented by $C(U_n)$. Similar to [1], $X^{\rho+\varepsilon}$. will denote this space when U_n is the interval of ordinals $[1, \rho + \varepsilon]$ provided with the order topology. $\mathcal{K}(X, Y)$ represents the Banach compact operator spaces from X to Y given Banach spaces X and Y.

The Banach space of all X-valued functions f defined on Ω is represented by $c_0(\Omega, X)$ given a fixed set Ω. This means that for every $\varepsilon > 0$, there exists a finite subset F_ε of Ω with $\| f(\xi - \varepsilon) \| < \varepsilon$ for all $\xi - \varepsilon \notin F_\varepsilon$, under the supremum norm, and $l_{q-\varepsilon}(\Omega, X). 1 \leq q - \varepsilon < \infty$, indicates the subset of $c_0(\Omega, X)$ made up of those f for which $\Sigma_{\xi-\varepsilon\in\Gamma} \| f(\xi - \varepsilon) \|^{q-\varepsilon} < \infty$. under the norm $\| f \| = (\Sigma_{\xi-\varepsilon\in\Gamma} \| f(\xi - \varepsilon) \|^{q-\varepsilon})^{1/q-\varepsilon}$. For the situation $X = R$, we also write $c_0(\Omega)$ and $l_{q-\varepsilon}(\Omega)$, as usual. When Ω is countably infinite, we just write c_0 and $l_{q-\varepsilon}$.

A comprehensive isomorphic classification of spaces $\mathcal{K}\big(C(Q), C(Q)\big)$. where Q is a countable compact metric space, is given by C. Samuel [17] in a recent publication. The traditional theorem of Mazurkiewicz and Sierpi´nski, asserts that every countable compact metric space is homeomorphic to an interval of ordinals $[1, \rho + \varepsilon]$ with $\omega \leq \rho + \varepsilon < \alpha + \varepsilon$, which is one of the primary instruments utilized in the demonstration of this claim. In fact, a comprehensive isomorphic categorization of spaces $\mathcal{K}(R^{\rho+\varepsilon}, R^{\rho+\varepsilon})$with $\omega \leq \rho + \varepsilon < \alpha + \varepsilon$ was provided in [17].

This is meant as an extension of [17]. We extend the classification of compact operator spaces to more general settings.

Our focus is specifically on the isomorphic classification of the spaces $\mathcal{K}(X, c_0(\Omega)^{\rho+\varepsilon})$, where X is a handy class of Banach spaces and $\rho + \varepsilon \geq \omega$.

This special case study offers an extension of the classical isomorphic classification of the spaces $R^{\rho+\varepsilon}, \rho + \varepsilon \geq \omega$, and a complete isomorphic classification of spaces $\mathcal{K}(R^{\xi}, R^{\rho+\varepsilon})$ with $\omega \leq \xi < \alpha + \varepsilon$ and $\rho + \varepsilon \geq \omega$, which naturally extends the previously mentioned result of C. Additionally, a complete isomorphism of some spaces $\mathcal{K}(X, Y^{\rho+\varepsilon})$, where $\rho + \varepsilon \geq \omega$ and X and Y are some of the spaces $l_{q-\varepsilon}(\Omega), 1 \leq q - \varepsilon < \infty, or\ c_0(\Omega)$ for some set Ω (Remark 4.1).

Next, we focus on the spaces $\mathcal{K}(X, c_0(\Omega)^{\rho+\varepsilon}), \rho + \varepsilon \geq \omega$. It is commonly known [3], [7] that $\mathcal{K}(X, c_0(\Omega)^{\rho+\varepsilon})$ is isomorphic to the injective tensor product $X^* \widehat{\otimes}_\varepsilon c_0(\Omega)^{\rho+\varepsilon}$, where X^* is the

dual space of X, since $c_0(\Omega)^{\rho+\varepsilon}$ possesses the approximation property [3]. Additionally, this final space is isomorphic to $c_0(\Omega, X^*)^{\rho+\varepsilon}$ by [3]. So as to determine the isomorphic classification of the spaces $c_0(\Omega, X^*)^{\rho+\varepsilon}$, it is necessary to hypothesize that X^* possesses the Mazur property.

A Banach space X is said to possess the Mazur Property (abbreviated MP) if each element of X^{**} , the sequentially weak* continuous bidual space of X, is also weak* continuous and, hence, an element of X. These spaces were examined in [4], [12], [9], and [19], where they were referred to as d-complete and μB-spaces, respectively. For some set Ω, the weakly compactly generated (WCG) spaces, and hence the spaces $l_{q-\varepsilon}(\Omega), 1 \leq q-\varepsilon < \infty$, and $c_0(\Omega)$ are included in the class of Banach spaces with the MP condition [4].

Recalling that the density feature dens X of a Banach space X is the smallest cardinal number δ such that there exists a set of cardinality δ dense in X allowed us to express our primary result. $\bar{\xi}$ will be used to represent the cardinality of an ordinal ξ. The symbol $|\Omega|$, will be used to indicate the cardinality of a set Ω. When X and Y are isomorphic to one another, we write $X \sim Y$; otherwise, we write $X \nsim Y$. Therefore, the goal is to demonstrate the following result for the isomorphic categorization of spaces $c_0(\Omega, X)^{\rho+\varepsilon}, \rho+\varepsilon \geq \omega$

Theorem 1.1. Suppose X is a Banach space that has two infinite ordinals $\varepsilon \geq 0$, a set Ω, an initial ordinal α, and no copies of c_0. Next:

(i) If $c_0(\Omega, X)^{\xi} \sim c_0(\Omega, X)^{\rho+\varepsilon}$ with $\bar{\xi}+\varepsilon = \aleph_1$ and X has MP, or $\bar{\xi}+\varepsilon > \aleph_1$ and $|\Omega| < \bar{\xi}+\varepsilon$ then $\bar{\xi} = \bar{\xi}+\varepsilon$.

(ii) Let's say that $\bar{\xi} = \bar{\xi}+\varepsilon = \bar{\alpha}$ and that $\alpha = \omega$, α is a singular ordinal with $|\Omega| < \bar{\alpha}$, $\varepsilon = 0, |\Omega| \leq \aleph_0, (\alpha+\varepsilon)^2 \leq \xi$ and X has MP, or α is a regular ordinal with $\varepsilon < 0$, $|\Omega|$dens $X^* < \bar{\alpha}, \alpha_2 \leq \xi$ and X has MP. Therefore, if and only if $\rho+\varepsilon < \xi^{\omega}$, $c_0(\Omega, X)^{\xi} \sim c_0(\Omega, X)^{\rho+\varepsilon}$.

(iii) Assume that α is α nondenumerable regular ordinal, such that $,\rho+\varepsilon \in [\alpha, \alpha^2]$, and let $\xi', \xi'+\varepsilon, \rho+\varepsilon, \delta$ be ordinals such that $= \alpha\xi' + \xi + \varepsilon, \rho+\varepsilon = \alpha(\xi'+\varepsilon) + \delta, \xi', \xi'+\varepsilon \leq \alpha, \rho+\varepsilon, \delta < \alpha$. Suppose that X possesses MP. If $\varepsilon = 0$, or $\varepsilon < 0$, and $|\Omega|$dens $X^* < \bar{\alpha}$, then $c_0(\Omega, X)^{\xi} \sim c_0(\Omega, X)^{\rho+\varepsilon}$ if and only if $|\Omega|\bar{\xi}' < \aleph_0$ and $X^{|\Omega|\bar{\xi}'} \sim X^{|\Omega|\bar{\xi}'+\varepsilon}$ or $\aleph_0 \leq |\Omega|\bar{\xi}' = |\Omega|\bar{\xi}'+\varepsilon$.

(iv) Let α be α nondenumerable regular ordinal, such that $\alpha \leq \xi < \alpha^2 \leq \rho+\varepsilon$, and let X have MP. If $\varepsilon = 0$ and $|\Omega| \leq \aleph_0$, or if $\varepsilon < 0$ and $|\Omega|$ dens $X^* < \bar{\alpha}$, then $c_0(\Omega, X)^{\xi} \nsim c_0(\Omega, X)^{\rho+\varepsilon}$.

Remark 1.2. It is evident that Theorem 1.1 offers a comprehensive isomorphic categorization of spaces $c_0(\Omega,X)^{\rho+\varepsilon}, \rho+\varepsilon \geq \omega, |\Omega| \leq \aleph_0$. In the event that X does not include a copy of c_0, X has MP and dens $X^* \leq \aleph_1$. When $X = R$ and Ω contains just one point, this conclusion is the result of Bessaga and Pełczy´nski [1], when $\omega \leq \rho+\varepsilon < \alpha+\varepsilon$; Semadeni [18], when $\alpha+\varepsilon \leq \rho+\varepsilon < (\alpha+\varepsilon)\omega$; and separately Kislyakov [10] and Gul'ko and Os'kin [8] in the general situation.

Remark 1.3. Since $(1+\varepsilon)_r$ indicates the least real-valued measurable cardinal, It is noted that if X has MP, then $l_1(\Omega,X)$ also has MP whenever $|\Omega| < (1+\varepsilon)_r$ [4]. Furthermore, if X does not contain a copy of c_0, then $l_1(\Omega,X)$ does not contain a copy of c_0, as we can demonstrate using a classic gliding hump argument; see [2].

Thus, using the well-known $c_0 - l_1 - l_\infty$ Theorem of Bessaga and Pełczy´ski [11], we can observe that Theorem 1.1 also offers a comprehensive isomorphic categorization of spaces under the Continuous Hypothesis, $2^{\aleph_0} = \aleph_1$. where $|\Omega| \leq \aleph_0$, $\mathbb{N}$ is the set of positive integers, X contains no complemented copy of l_1, X^* has MP and dens $X^{**} \leq \aleph_1$, and $\mathcal{K}\left(X^\xi, c_0(\Omega)^{\rho+\varepsilon}\right) \sim c_0\left(\Omega, l_1(\mathbb{N},X^*)\right)^{\rho+\varepsilon}, \omega \leq \xi < \alpha+\varepsilon, \rho+\varepsilon \geq \omega$. Note that if $X = R$, Ω contains a single point, and $\omega \leq \xi = \rho+\varepsilon < \alpha+\varepsilon$, the Continuous Hypothesis is not required. This is precisely what C. Samuel [17] had in mind when he inspired this work.

2. Some preliminary results

As a result of certain lemmas, the proof of Theorem 1.1 will be shown. Consider Banach spaces X and Y. When we say that $X \hookrightarrow Y$, we indicate that Y has a subspace that is isomorphic to X. First, observe that we obtain the following outcome by using the same steps as in the proof of [17].

Lemma 2.1. Assume $\rho+\varepsilon$ is an infinite ordinal and X is a Banach space. If $R^{(\rho+\varepsilon)^\omega} \hookrightarrow X^{\rho+\varepsilon}$, then for any ordinal $\varepsilon > 0$, $R^{\rho+\varepsilon} \hookrightarrow X^{\xi-\varepsilon}$.

Lemma 2.2. Assume that $R^{\omega^\omega} \hookrightarrow c_0(\Omega,X)$ for some set Ω, where X is a Banach space. Consequently, $c_0 \hookrightarrow X$.

Proof. It is simple to demonstrate that there exists $\Omega_1 \subset \Omega$ with $|\Omega_1| = \aleph_0$ such that $R^{\omega^\omega} \hookrightarrow c_0(\Omega_1,X)$ since R^{ω^ω} is separable. Moreover, $c_0(\Omega_1,X) \sim X^\omega$ by [1]. Thus, according to Lemma 2.1, for any $0 \leq \varepsilon < \omega - 1$, $R^\omega \hookrightarrow X^{1+\varepsilon}$. Therefore, $c_0 \sim R^\omega \hookrightarrow X$ in accordance with [15] and [1] once more. □

Lemma 2.3. Assume α is an infinite initial ordinal and X is a Banach space. Consider a set Ω with $|\Omega| < \bar{\alpha}$ and $\rho+\varepsilon < \alpha$, such that $R^\alpha \hookrightarrow c_0(\Omega,X)^{\rho+\varepsilon}$. Consequently, $c_0 \hookrightarrow X$.

Proof. Assume that I is the collection of isolated points in $[1,\alpha]$. Consequently, $c_0(I) \hookrightarrow R^{\alpha}$, and because $c_0(\Omega,X)^{\rho+\varepsilon} \sim c_0(\Omega) \widehat{\otimes} R^{\rho+\varepsilon} \widehat{\otimes} X \sim C(\xi-\varepsilon)(\Omega) \widehat{\otimes} R^{\rho+\varepsilon} \widehat{\otimes} X \sim C(\xi-\varepsilon)(\Omega \times [1,\rho+\varepsilon],X)$, on a subspace of $C(\xi-\varepsilon)(\Omega \times [1,\rho+\varepsilon],X)$, where $(\xi-\varepsilon)\Omega$ is the Alexandroff compactification of the discrete set Ω, there is an isomorphism T from $c_0(I)$. Since $M \in (0,+\infty)$, let $M \leq \| (x) \|, \forall x \in c_0(I), \| x \| = 1$. Shown by $(e_i)_{i\in I}$, this is the unit-vectors basis of $c_0(I)$; for any $i,j \in I, e_i(j) = 1$ if $i = j$ and $e_i(j) = 0$ if $i \neq j$. Assume that H is the collection of isolated points of $(\xi-\varepsilon)\Omega \times [1,\xi-\varepsilon]$.. Theoretically, then, $| H | < \bar{\alpha}$. $I_h = \{i \in I : M/2 \leq \| T(e_i)(h) \|\}$ is defined for a given $h \in H$. Consequently, $I = \bigcup_{h\in H} I_h$.

As a result, $h \in H$ that satisfies $| I_h | \geq \aleph_0$ exists. The subspace of $c_0(I)$ that contains the elements f such that $f(\xi-\varepsilon) = 0$ for any $\xi-\varepsilon \notin I_h$. is what we designate as $c_0(I_h)$. Consider the natural projection as $(q-\epsilon)_h : C((\xi-\varepsilon)\Omega \times [1,\rho+\varepsilon],X) \to X$, where $(q-\varepsilon)_h(f) = f(h)$, for all $f \in ((\xi-\varepsilon)\Omega \times [1,\rho+\varepsilon],X)$. The operator $L = (q-\varepsilon)_h T_{|_{c_0}}(I_h) : c_0(I_h) \to X$ is then examined. Inf $\{\| (e_i) \| : i \in I_h\} > 0$ then appears. Therefore, in accordance with [13], $J \subset I_h$ with $| J | = | I_h |$ exists where $L_{|_{c_0}(J)}$ is an isomorphism onto its image. □

Lemma 2.4. Assume that X is a Banach space then $c_0(I) \hookrightarrow c_0(J,X)$ for specific sets I and J, where $| J | < | I |$ and $| I | \geq \aleph_0$. Consequently, $c_0 \hookrightarrow X$.

We must recall some Banach spaces that were presented in [5] in order to assert the following lemma. Suppose that $\xi-\varepsilon$ is an ordinal. The notation for A $(\xi-\varepsilon)$ -sequence in a set A is $(x_\theta)_{\theta<\xi-\varepsilon}$ and is a function $f: [1,\xi-\varepsilon) \to A$. Assume that X is a Banach space, $\epsilon < 0$, φ is any ordinal, and α is a regular ordinal. The space of all $x^{**} \in X^{**}$ will be denoted by X_α^φ, which has the following property: for every limit ordinal $\mathcal{B} < \alpha$ and for every φ-sequence $x^{\rho+\varepsilon} = (x_\xi^*(\rho+\varepsilon))_{\xi<\mathcal{B}}$ of B-sequences of X^* such that there exists $M \in R$ with $\| x_\xi^*(\rho+\varepsilon) \| \leq M, \forall \rho+\varepsilon < \varphi, \forall \xi < \mathcal{B}$, and such that $x_\xi^*(\rho+\varepsilon)(x) \xrightarrow{\xi\to\mathcal{B}} 0, \forall x \in X$, uniformly in $\rho+\varepsilon$, we have $x^{**}(x_\xi^*(\rho+\varepsilon)) \xrightarrow{\xi\to\mathcal{B}} 0$ uniformly in $\rho+\varepsilon$. The canonical image of X in X^{**} is indicated by cX if X is a Banach space.

The following is easily proven by applying the Hahn-Banach theorem, as in the proof of [10]:

Lemma 2.5. Given a regular ordinal α and any ordinal φ, consider X is a closed subspace of the Banach space. Then, $\frac{X_\alpha^\varphi}{cX} \hookrightarrow \frac{Y_\alpha^\varphi}{cY}$.

The Scholar Journal for Sciences & Technology sjst

Lemma 2.6. Suppose that X is a Banach space with MP, a set Ω, and a nondenumerable regular ordinal α. Consider the following scenarios: $\varepsilon = 0$ and $|\Omega| \leq \aleph_0$, or $\varepsilon < 0$ and $|\Omega|$ dens $X^* < \bar{\alpha}$. For a given $\rho + \varepsilon < \alpha^2$, $c_0 \hookrightarrow X$ if $R^{\alpha^2} \hookrightarrow c_0(\Omega, X)^{\rho+\varepsilon}$.

Proof. Two cases will be distinguished.

Case 1. $\rho + \varepsilon < \alpha$. In this instance, $R^{\alpha} \hookrightarrow c_0(\Omega, X)^{\rho+\varepsilon}$ and Lemma 2.3 imply that $c_0 \hookrightarrow X$.

Case 2. $\alpha \leq \rho + \varepsilon < \alpha^2$. therefore for certain ordinals $\xi < \alpha$ and $\theta < \alpha$, $\rho + \varepsilon = \alpha\xi + \theta$. Given that $R^{\rho+\varepsilon} \sim R^{\alpha\xi}$ [10], we have

$$R^{\alpha^2} \hookrightarrow c_0(\Omega, X)^{\rho+\varepsilon} \sim c_0(\Omega) \widehat{\otimes} R^{\rho+\varepsilon} \widehat{\otimes} X \sim c_0(\Omega) \widehat{\otimes} R^{\alpha\xi} \widehat{\otimes} X \sim c_0(\Omega, X)^{\alpha\xi}.$$

Pick φ the ordinal defined by

$$\varphi = \begin{cases} \omega \ \text{if } \varepsilon = 0\,, \\ \text{the first ordinal of cardinality } |\Omega| \text{ dens } X^* \text{ if } \varepsilon < 0\ . \end{cases}$$

Consider two sets, I and J, such that $|\, I \,| = \bar{\alpha}$ and $|\, J \,| = \bar{\xi}$. Based on [5] and [6], we deduce that

$$c_0(I) \sim \frac{(R^{\alpha^2})_{\alpha}^{\varphi}}{cR^{\alpha^2}} \hookrightarrow \frac{(c_0(\Omega, X)^{\alpha\xi})_{\alpha}^{\varphi}}{cc_0(\Omega, X)^{\alpha\xi}} \sim c_0(J, c_0(\Omega, X)).$$

Therefore, Lemma 2.4 is sufficient to infer that $c_0 \hookrightarrow X$. □

Lemma 2.7. Assume that α is an initial ordinal and that $\varepsilon \geq 0$ ordinals with $\bar{\xi} = \bar{\xi} + \varepsilon = \bar{\alpha}$. Put $\alpha_0 = \omega^{\omega}$ if $\alpha = \omega, \alpha_0 = \alpha$ if α is a single ordinal, and $\alpha_0 = \alpha_2$ otherwise. If α is a regular ordinal, then $\varepsilon < 0$. Consider X as a Banach space with MP that does not include a copy of c_0, and Ω as a set satisfying

(i) Ω is arbitrary if $\alpha = \omega$,

(ii) $|\Omega| < \bar{\alpha}$ if α is a singular ordinal,

(iii) $|\Omega| \leq \aleph_0$ if $\varepsilon = 0$,

(iv) $|\Omega|$ dens $X^* < \bar{\alpha}$ if α is a regular ordinal with $\varepsilon < 0$.

If $R^{\rho+\varepsilon} \hookrightarrow c_0(\Omega, X)^{\xi}$ with $\alpha_0 \leq \xi$, then $R^{\rho+\varepsilon} \hookrightarrow R^{\xi}$.

Proof. We provide two ordinal sets

$I_1 = \{\theta : \bar{\theta} = \bar{\alpha}, \alpha_0 \leq \theta, R^{\theta} \nrightarrow R^{\xi-\varepsilon}, \forall(\xi - \varepsilon) < \theta\}$.

$I_2 = \{\theta : \bar{\theta} = \bar{\alpha}, \alpha_0 \leq \theta, R^{\theta} \nrightarrow c_0(\Omega, X)^{\xi-\varepsilon}, \forall(\xi - \varepsilon) < \theta\}$.

First, we'll demonstrate that $I_1 = I_2$. Obviously, $I_1 \subset I_2$. It is evident that $\alpha_0 \in I_2$ based on Lemmas 2.2, 2.3, and 2.6. At this point, suppose that I_2 is a proper subset of I_1. Consider α_1 to be

the least of $I_1 \setminus I_2$. We possess $\alpha_0 < \alpha_1$. Given that $\alpha_1 \notin I_2$, there is an ordinal $\xi_1 - \varepsilon < \alpha_1$ such that $R^{\alpha_1} \hookrightarrow c_0(\Omega, X)^{\xi_1 - \varepsilon}$.

Assume that $\alpha_2 = min\{\xi - \varepsilon, \alpha_0 \leq \xi - \varepsilon < \alpha_1 : R^{\alpha_1} \hookrightarrow c_0(\Omega, X)^{\xi - \varepsilon}\}$. We possess $\alpha_2 \leq \xi_1 - \varepsilon$. We shall now demonstrate that $\alpha_2 \in I_1$. If this isn't the case , then $R^{\alpha_2} \hookrightarrow R^{\xi_2 - \varepsilon}$ exists for an ordinal $\xi_2 - \varepsilon < \alpha_2$. Accordingly,

$$c_0(\Omega, X)^{\alpha_2} \sim c_0(\Omega) \widehat{\otimes} R^{\alpha_2} \widehat{\otimes} X \hookrightarrow c_0(\Omega) \widehat{\otimes} R^{\xi_2 - \varepsilon} \widehat{\otimes} X \sim c_0(\Omega, X)^{\xi_2 - \varepsilon}$$

This means that $R^{\alpha_1} \hookrightarrow c_0(\Omega, X)^{\xi_2 - \varepsilon}$ is not what α_2 is defined to be. As a result, $\alpha_2 \in I_1$, and since $\alpha_2 < \alpha_1$, according to the definition of α_1, $\alpha_2 \in I_2$. That is, $R^{\alpha_2} \nrightarrow c_0(\Omega, X)^{\xi - \varepsilon}$, $\forall(\xi - \varepsilon) < \alpha_2$. Lemma 2.1 thus leads us to the conclusion that $R^{\alpha_2^{\omega}} \nrightarrow c_0(\Omega, X)^{\alpha_2}$.

However, take notice that by [10], $R^{\alpha_1} \sim R^{\alpha_2}$, which is irrational by the definition of α_1, if $\alpha_1 < \alpha_2^{\omega}$. Thus, $R^{\alpha_2^{\omega}} \hookrightarrow R^{\alpha_1}$, and $\alpha_2^{\omega} \leq \alpha_1$. Additionally, according to the definition of $\alpha_2, R^{\alpha_1} \hookrightarrow c_0(\Omega, X)^{\alpha_2}$. Consequently, $R^{\alpha_2^{\omega}} \hookrightarrow c_0(\Omega, X)^{\alpha_2}$, which is in opposition to what we have just demonstrated. Thus, $I_1 = I_2$. Then, to finish the lemma's proof, assume that $R^{\rho+\varepsilon} \nrightarrow R^{\xi}$ and let $\xi_1 = min\{\theta : R^{\rho+\varepsilon} \hookrightarrow R^{\theta}\}$. Therefore, $R^{\xi_1} \nrightarrow R^{\xi - \varepsilon}$, $\forall(\xi - \varepsilon) < \xi_1$, and $\xi < \xi_1 \leq \rho + \varepsilon$. In particular,

$R^{\xi_1} \hookrightarrow R^{\rho+\varepsilon} \hookrightarrow c_0(\Omega, X)^{\xi}$, which makes it irrational that $\xi_1 \in I_1 = I_2$. □

3. Proof of Theorem 1.1

(i) First, suppose that X has MP and $\bar{\xi} + \varepsilon = \aleph_1$. Consider $\bar{\xi} = \aleph_0$. Then $R^{\alpha+\varepsilon} \hookrightarrow R^{\rho+\varepsilon} c_0(\Omega, X)^{\xi}$, which is irrational since $R^{\alpha+\varepsilon}$ does not have MP [19] and $c_0(\Omega, X)^{\xi}$ has MP [9].

Assume now that $|\Omega| < \bar{\xi} + \varepsilon$ and $\bar{\xi} + \varepsilon > \aleph_1$. Consider that α is the initial ordinal of cardinality $\bar{\xi} + \varepsilon$ and assume that $\bar{\xi} < \bar{\xi} + \varepsilon$. Next,

This is contradicted by Lemma 2.3, which states that $c_0 \hookrightarrow X$, and $R^{\alpha} \hookrightarrow R^{\rho+\varepsilon} \hookrightarrow c_0(\Omega, X)^{\xi}$.

To verify the sufficiency of statements (ii) and (iii) it suffices to keep in mind [10] and notice that if $R^{\xi} \sim R^{\rho+\varepsilon}$, then

$$c_0(\Omega, X)^{\xi} \sim c_0(\Omega) \widehat{\otimes} X^{\xi} \sim c_0(\Omega) \widehat{\otimes} X \widehat{\otimes} R^{\xi} \sim c_0(\Omega) \widehat{\otimes} X \widehat{\otimes} R^{\rho+\varepsilon} \sim c_0(\Omega) \widehat{\otimes} X^{\rho+\varepsilon}$$
$$\sim c_0(\Omega, X)^{\rho+\varepsilon}.$$

We then demonstrate that statements (ii) and (iii) are necessary.

(ii) First, suppose that $C(\xi - \varepsilon)(\Omega, X)^{\rho+\varepsilon} \sim C(\xi - \varepsilon)(\Omega, X)^{\xi}$. There are two cases that we distinguish:

Case 1. $\alpha = \omega$. According to the hypothesis, $R^{\rho+\varepsilon} \hookrightarrow c_0(\Omega, X)^{\xi}$ if $\xi \geq \omega^{\omega}$

The Scholar Journal for Sciences & Technology sjst

Therefore, $R^{\rho+\varepsilon} \hookrightarrow R^{\xi}$ is implied by Lemma 2.7, and by [10] we have $\rho + \varepsilon < \xi^{\omega}$. However, we deduce that $R^{\xi} \sim R^{\omega}$ if $\xi < \omega^{\omega}$, again by [10]. Assume that Ω_1 is a set of cardinality $|\Omega|\ \aleph_0$. Thus,

$c_0(\Omega_1, X) \sim c_0(\Omega, X)^{\omega} \sim c_0(\Omega_1, X)^{\xi} \sim c_0(\Omega_1, X)^{\rho+\varepsilon}$.

Then, $R^{\omega^{\omega}} \hookrightarrow R^{\rho+\varepsilon} \hookrightarrow c_0(\Omega_1, X)$ would exist if $\rho + \varepsilon > \xi^{\omega} \geq \omega^{\omega}$.

Consequently, Lemma 2.2 would suggest that $c_0 \hookrightarrow X$, which is irrational.

Case 2. $\alpha > \omega$. $R^{\rho+\varepsilon} \hookrightarrow c_0(\Omega, X)^{\rho+\varepsilon} \sim c_0(\Omega, X)^{\xi}$ if $\rho + \varepsilon > \xi^{\omega}$. Since Lemma 2.7's assumptions are true, we arrive at the nonsensical conclusion that $R^{\rho+\varepsilon} \hookrightarrow R^{\xi}$ [10].

(iii) Suppose that I and J are two sets such that $|I| = \bar{\xi}'$ and $|J| = \bar{\xi}' + \varepsilon$. Let φ be as in Lemma 2.6's proof. According to [5] and [6],

$$c_0\big(I, c_0(\Omega, X)\big) \sim \frac{\left(c_0(\Omega,X)^{\alpha\xi'}\right)_{\alpha}^{\varphi}}{cc_0(\Omega,X)^{\alpha\xi'}} \sim \frac{\left(c_0(\Omega,X)^{\alpha\xi'+\varepsilon}\right)_{\alpha}^{\varphi}}{cc_0(\Omega,X)^{\alpha\xi'+\varepsilon}} \sim c_0(J, c_0(\Omega, X)).$$

Therefore, $X^{|\Omega|\,\bar{\xi}'} \sim X^{|\Omega|\bar{\xi}'+\varepsilon}$ if $|\Omega|\bar{\xi}'$ and $|\Omega|\ \bar{\xi}' + \varepsilon$ are finite.

However, using Lemma 2.4, we can see that $|\Omega|\ \bar{\xi}' = |\Omega|\bar{\xi}' + \varepsilon$ if $|\Omega|\ \bar{\xi}'$ or $|\Omega|\ \bar{\xi}' + \varepsilon$ is infinite.

(iv) With $\alpha \leq \xi < \alpha^2 \leq \rho + \varepsilon$, suppose that $c_0(\Omega, X)^{\xi} \sim c_0(\Omega, X)^{\rho+\varepsilon}$. Consequently, $R^{\alpha^2} \hookrightarrow c_0(\Omega, X)^{\xi}$. Therefore, $c_0 \hookrightarrow X$ by Lemma 2.6. This completes Theorem 1.1's proof.

4. Final remarks and some problems

Remark 4.1. Assume that I and J are two sets and that $1 < q - \varepsilon < \infty, 1 < q < \infty$.

Notably, the approximation property is present in the spaces $l_{q-\varepsilon}(I), c_0(I)$, and $l_1(I)$ [3].

Furthermore, a copy of c_0 is absent from the spaces $l_1(I) \,\widehat{\otimes}\, l_1(J), l_{q-\varepsilon}(I) \,\widehat{\otimes}\, l_1(J)$, and $l_{q-\varepsilon}(I) \,\widehat{\otimes}\, l_q(J)$ with $q' - \varepsilon > q$, where $\frac{1}{q-\varepsilon} + \frac{1}{q'-\varepsilon} = 1$ [16].

In conclusion, MP is present in the spaces $l_{q-\varepsilon}(I) \,\widehat{\otimes}\, l_q(J), l_{q-\varepsilon}(I) \,\widehat{\otimes}\, l_1$ and $l_1 \,\widehat{\otimes}\, l_1$ [9].

Therefore, the following spaces are completely isomorphically classified in Remarks 1.2 and 1.3:

(1) $\mathcal{K}(l_{q-\varepsilon}(I), c_0(J)^{\rho+\varepsilon})$, where $\rho + \varepsilon \geq \omega, |\ I\ | \leq \aleph_1$ and $|\ J\ | \leq \aleph_0$.

(2) $\mathcal{K}(c_0(I), c_0(J)^{\rho+\varepsilon})$, where $\rho + \varepsilon \geq \omega, |\ I\ | \leq \aleph_0$ and $|\ J\ | \leq \aleph_0$.

(3) $\mathcal{K}(l_{q-\varepsilon}(I), l_q(J)^{\rho+\varepsilon})$, where $\rho + \varepsilon \geq \omega, |\ I\ | \leq \aleph_1, |\ J\ | \leq \aleph_1$ and $q - \varepsilon > q$.

(4) $\mathcal{K}(c_0(I), l_{q-\varepsilon}(J)^{\rho+\varepsilon})$, where $\rho + \varepsilon \geq \omega, |\ I\ | \leq \aleph_0$ and $|\ J\ | \leq \aleph_1$.

(5) $\mathcal{K}\big(l_{q-\varepsilon}(I), l_1(J)^{\rho+\varepsilon}\big), where\ \rho + \varepsilon \geq \omega, |\ I\ | \leq \aleph_1\ and\ |\ J\ | \leq \aleph_0$.

(6) $\mathcal{K}(c_0(I), l_1(J)^{\rho+\varepsilon}), where\ \rho + \varepsilon \geq \omega, |\ I\ | \leq \aleph_0\ and\ |\ J\ | \leq \aleph_0$.

Naturally, Remark 4.1 causes the following problems:

The Scholar Journal for Sciences & Technology sjst

Problem 4.2. Classify the following spaces up to an isomorphism:

(1) $\mathcal{K}(l_1, c_0^{\rho+\varepsilon}), where\ \rho+\varepsilon \geq \omega.$

(2) $\mathcal{K}\left(l_1, l_{q-\varepsilon}^{\rho+\varepsilon}\right), where\ \rho+\varepsilon \geq \omega\ and\ 1 \leq q-\varepsilon < \infty.$

(3) $\mathcal{K}(l_{q-\varepsilon}, l_q^{\rho+\varepsilon}), where\ \rho+\varepsilon \geq \omega$ and $1 < q-\varepsilon \leq q < \infty$. Lastly,

the Stone-Cech compactification of $\mathbb{N}$ is shown by $\mathcal{B}\mathbb{N}$. The existence of a continuous function of $\mathcal{B}\mathbb{N}$ onto $\mathcal{B}\mathbb{N} \times \mathcal{B}\mathbb{N}$ implies that $C(\mathcal{B}\mathbb{N}) \,\widehat{\otimes}\, C(\mathcal{B}\mathbb{N}) \sim C(\mathcal{B}\mathbb{N} \times \mathcal{B}\mathbb{N}) \hookrightarrow C(\mathcal{B}\mathbb{N})$, and consequently, $c_0(\mathbb{N}, C(\mathcal{B}\mathbb{N})) \,\widehat{\otimes}\, c_0(\mathbb{N}, C(\mathcal{B}\mathbb{N})) \hookrightarrow c_0(\mathbb{N}, C(\mathcal{B}\mathbb{N}))$. Additionally, for any $\alpha \geq \alpha+\varepsilon$, $R^\alpha \nrightarrow C(\mathcal{B}\mathbb{N})$; otherwise, for some set Ω with $|\Omega| = \aleph_1$, $c_0(\Omega) \hookrightarrow C(\mathcal{B}\mathbb{N})$, which is absurd as $\mathcal{B}\mathbb{N}$ fulfills the Countable Chain Condition [14]. Thus, Problem 4.2(1) is partially solved by the following proposition.

Proposition 4.3. Assume that $\varepsilon \geq 0$ and that α is an infinite initial ordinal with $\bar{\xi} = \bar{\xi}+\varepsilon = \bar{\alpha}$. For any $0 \leq \varepsilon < \omega-1$ and $X^\xi \sim X^{\rho+\varepsilon}$, if X is a Banach space that satisfies $X \,\widehat{\otimes}\, X \hookrightarrow X, R^\alpha \nrightarrow X^{1+\varepsilon}$, then $\rho+\varepsilon < \xi^\omega$.

Proof. Let's say $\rho+\varepsilon \geq \xi^\omega$. After that, $R^{\xi^\omega} \hookrightarrow X^\xi$. Let

$$\xi_1 = min\{\xi-\varepsilon : \varepsilon > 0, R^{\xi^\omega} \hookrightarrow X^{\xi-\varepsilon}\}.$$

Given that ξ_1 must be infinite according to the hypothesis and that $\xi_1 \leq \xi$, we may conclude that $R^{\xi_1^\omega} \hookrightarrow R^{\xi^\omega} \hookrightarrow X^{\xi_1}$. Thus, for some $\varepsilon > 0$, $R^{\xi_1} \hookrightarrow X^{\xi_1-\varepsilon}$ in accordance with Lemma 2.1. As a result,

$$X^{\xi_1} \sim R^{\xi_1} \,\widehat{\otimes}\, X \hookrightarrow X^{\xi_1-\varepsilon} \,\widehat{\otimes}\, X = (X \,\widehat{\otimes}\, X)^{\xi_1-\varepsilon} \hookrightarrow X^{\xi_1-\varepsilon}.$$

Therefore, by definition of ξ_1, $R^{\xi_1^\omega} \hookrightarrow X^{\xi_1} \hookrightarrow X^{\xi_1-\varepsilon}$ is absurd. □

The Scholar Journal for Sciences & Technology sjst

References

1. Bessaga, C., & Pełczyński, A. (1960). Spaces of continuous functions (IV).(On isomorphical classification of spaces of continuous functions). Studia Mathematica, 19(1), 53-62.

2. Burlando, L. (1988). On subspaces of direct sums of infinite sequences of Banach spaces. Dipt., Univ..96 – 105.

3. Diestel, J., & Uhl, J. J., Jr. (1977). Vector measures (Vol. 15). American Mathematical Society.

4. Edgar, G. A. (1979). Measurability in a Banach space, II. Indiana University Mathematics Journal, 28(4), 559-579.

5. Galego, E. (1999). How to generate new Banach spaces non-isomorphic to their cartesian square. Bulletin of the Polish Academy of Sciences. Mathematics, 47(1), 22-25.

6. Galego, E. M. (2001). Banach spaces of continuous vector-valued functions of ordinals. Proceedings of the Edinburgh Mathematical Society, 44(1), 49-62.

7. Grothendieck, A., & Grothendieck, A. (1955). Produits tensoriels topologiques et espaces nucléaires (Vol. 16, p. 336). Providence: American Mathematical Society.

8. Gul'ko, S. P., & Os' KiN, A. V. (1975). Isomorphic classification of spaces of continuous functions on totally ordered compact sets. Functional Analysis and Its Applications, 9(1), 56-57.

9. Kappeler, T. (1986). Banach spaces with the condition of Mazur. Mathematische Zeitschrift, 191, 623-631.

10. Kislyakov, S. V. E. (1975). Classification of spaces of continuous functions of ordinals. Siberian Mathematical Journal, 16(2), 226-231.

11. Lindenstrauss, J., & Tzafriri, L. (1977). Classical Banach Spaces. Pt. 1. Sequence Spaces. Springer.

12. Leung, D. H. (1991). On Banach spaces with Mazur's property. Glasgow Mathematical Journal, 33(1), 51-54.

13. Rosenthal, H. (1970). On relatively disjoint families of measures, with some applications to Banach space theory. Studia Mathematica, 37(1), 13-36.

14. Rosenthal, H. P. (1970). On injective Banach spaces and the spaces $L^{\infty}(\mu)$ for finite measures μ.

15. Samuel, C. (1979). Sur l_a reproductibilite des espaces l_p. Séminaire d'Analyse fonctionnelle (dit" Maurey-Schwartz"), 1-17.

16. Samuel, C. (1980). Sur les sous-espaces de. Mathematica Scandinavica, 247-250.

17. Samuel, C. (2009). On spaces of operators on $C(Q)$ spaces (Q countable metric space). Proceedings of the American Mathematical Society, 137(3), 965-970.

18. Semadeni, Z. (1960). Banach spaces non-isomorphic to their Cartesian squares. II. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys, 8, 81-84.

19. Wilansky, A. (1981). Mazur spaces. International Journal of Mathematical Sciences, 4, 39–53.

20. Galego, E. (2009). On isomorphic classifications of spaces of compact operators. Proceedings of the American Mathematical Society, 137(10), 3335-3342.