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FULL PAPER

Features Toeplitz Operator on Herglotz Wave Functions that are Flexible

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Abstract

In a vacuum, Herglotz waves exist in$\mathbb{R}^2$. All solutions to the Helmholtz issue fall into the following categories: For the problem, we employ the Fourier transform. A circle with density $L^2(\mathcal{S}^1)$ is examined in $\mathbb{R}^2$ for its density function. We represent this space in terms of a Hilbert space, which has a similar structure.We examine the use of plane wave superpositions, or Herglotz wave functions. The function we want to approximate is thought of as a source distribution. Transform the field generated by the source distribution when the difference is measured in a ball. The Toeplitz operators are being examined in this paper with regard to several positive and limited vector variables. We will examine the compressible Toeplitz operator of come will explore the compressible Toeplitz operator and its properties, using Schatten as an example.The functions in are plane waves overlapping on one another. We show that the problem can be solved by looking at the function we want to converge as a set of sources and spinning the resulting field at the point where the difference can be found in a ball.

***Keywords*:** orthogonal, Carlson, Lebesgue, Hilbert, symmetric, operator, transform, harmonic, Herglotz, and spaces.

1. Introduction

In complex analytical theory, Toeplitz operators in Bergman spaces for symmetric functions have long been a key concept. For example, the Toeplitz operators help us determine which symbols have compact Toeplitz operators or are limited to certain integers. They are useful for holomorphic functions in Bergman spaces, harmonic functions, and kernel creation spaces [1,8, 9,12].

$$\Delta u + u = 0 \tag{1}$$

It can be expressed using $\mathbb{R}^n$ with the Fourier transform for $\mathbb{R}^n$ with an assessϕ, with λ indicating the Lebesgue measure at $\mathcal{S}^{n-1}$ and$\phi \in L^2(\mathcal{S}^{n-1})$. Additionally, define this is Herglotz wave operates, like

$$u(x) = \int_{\mathcal{S}^{n-1}} e^{ix.\omega}\phi(\omega)d\lambda(\omega) \tag{2}$$

The Herglotz wave functions have been described in a variety of ways ([10], [11], and [13]). For instance, the prerequisites the function of a Herglotz wave has necessary criteria as

$$\lim_{R\to\infty}\left(\frac{1}{R}\int_{|x|<R}|u(x)|^2dx\right)^{\frac{1}{2}} < \infty. \tag{3}$$

In respect to$\mathcal{W}^2$, the Herglotz space functions at$\mathbb{R}^n$
The harmonic relationship shows that Herglotz wave functions affect how sounds travel over obstacles and uneven surfaces. This is because certain Herglotz length functions show whether certain far-field patterns are dense in$L^2(\mathcal{S}^{n-1})$. This means that is a Helmholtz formula may be solved using the entire function$u$ in$\mathbb{R}^n$, is responsible for the dispersion of elastic waves. [3].The map in Equation (2), known as the expanded operator, provides an equivalent asymmetry $L^2(\mathcal{S}^{n-1})$ over$\mathcal{W}^2$, while that region $\mathcal{W}^2$ in Equation (3), this a Hilbert region.

There are expanded operator $\phi \to \widehat{\phi d\vartheta}$ displays the Helmholtz equation's full result for$u$. Since $n = 2$ as shown in [7], this operator is an isomorphism from $L^2(\mathcal{S}^1)$ to space.

$$\|u\|^2 = \int_{\mathbb{R}^2}(|u(x)|^2 + |\partial_\vartheta u(x)|^2)\mathcal{W}(x)dx \tag{4}$$

Assuming $\mathcal{W}(x) = \frac{1}{1+|x|^3}.\partial_\vartheta u = \nabla.\frac{x^\perp}{|x|}.x^\perp = (-x_2, x_1)$ while the Lebesgue measure in $\mathcal{S}^1$ is to be $d\vartheta$ in [5]. This allows us to learn more about the Herglotz wave functions in $\mathbb{R}^2$ from the Helmholtz solution set. Equation (3), which is equal to Equation (4). We also discover that the space of all completely functions so that$\|u\| < \infty$.
We can verify through every single one of these that the outcome is in [8], indicating that W^2 is in$\mathbb{R}^2$.
It is possible to extend the $\mathcal{W}^2$ components, like the Newman sequence,

$$u(x) = \sum_{n\in\mathbb{Z}} a_n J_n(r)e^{in\vartheta} \tag{5}$$

Assuming $x = re^{i\vartheta} = r(cos\vartheta + isin\vartheta)$ while $\sum_{n\in\mathbb{Z}}|a_n|^2 < \infty$, the Bessel function for putting that is represented by $J_n(r)$ [11].
We can think of the $\mathcal{W}^2$ space, which is the same as the Hilbert space of$L^2(\mathcal{S}^1)$, as a space after we have disrupted its kernel, which was talked about in [4, 5, 7].

When H^2 comprises space containing every distribution $(\mathbb{R}^2\backslash\{0\})$ so that$u$ — ^{2}), it is possible to write $L^2($ for $L^2(\mathbb{R}^2\backslash\{0\}, \mathcal{W}dx)$ when —— where$r$ 0.
H^2 As I indicated at the rule, $\mathcal{W}^2$ defines the component space H^2 that connects $\mathbb{R}^2$ so that the Helmholtz equation and Equation (4) is a Hilbert space.

$$\|u\|_{\mathcal{W}}^2 \quad \sum |a_n|^2 \sim \|\phi\|^2{}_{2(\mathcal{S}^1)}$$

Wheneverϕ $\sum$.
Indicate that$F$ $(r$ $)$, so that $\{e_n\}$ with$e$ $/\|F_n\|_{\mathcal{W}^2}$. The orthogonal basis represents toW^2. Let $\|F\|$, in [11].

$$^2 \quad ^2 \quad ^2 \quad {}_1 \quad ^2 \quad ^2 \quad — \quad 1. \tag{6}$$

It can take$P$ ^{2} $\mathcal{W}^2$.
It projects orthogonally. With and$y$ e , its proliferation kernel to $\mathcal{W}^2$ is$\mathcal{K}$ $\sum$ $)\overline{e\ (y)}$ $\sum$ ——————.). Is a genuine kernel, and

$$\langle u \quad)\rangle \quad {}_{\mathbb{R}^2}\left(\mathcal{K} \quad)\right)\mathcal{W} \quad dy.$$

If that$\|\mathcal{K}$ $)\|_{\mathcal{W}}^2$ $\sum$ —— 1, hence,

$$\sum \quad ^2 \tag{7}$$

Toeplitz operators were first talked about in the classic Holomorphic numbers. They can be defined asT_{ρ_j}. Toeplitz operators can be thought of as $\mathcal{W}^2$ with radius signs that are not positiveρ_j.

$$P(\rho_j u)$$

If the function ρ_j given as $[0,\infty)$ has a Carleson sign, then ^{2} $\mathcal{W}_{\rho_j}^2$ has continuously. While $\mathcal{W}_{\rho_j}^2$ can also be given by substituting the dimension as whereρ $)dx$, at the simplification of$\mathcal{W}^2$, then we are able to analyze the consistency Toeplitz operator.
A Toeplitz operator T_{ρ_j} has limitations in$\mathcal{W}^2$, and if ρ has a Carlson sign, then it defines the sequence's boundaries. Let me show that a non-negative$\rho_j \in L^1([0$).

$$\left\{(\quad ^2)\int \quad ^2 \quad dr\right\}$$

There are primary finding provides sufficient criteria at a combination of signs related to a set from one parameter at rational points that do not involve special functions. It was discovered in [2,6] that the weights suitable for Helmholtz formulas, our team wishes to substitute in this description, the Basel functions are to conceal the geometrical indicating in the Carlson signs.

Definition 1: With a circular function $\rho_j = \rho_j(|x| \geq 0)$ and$\Omega = \mathbb{R}^2\backslash\{0\}$, we may define a Toeplitz operator using

$$T_{\rho_j}u(x) = P(\rho_j u)(x) = \int_{\mathbb{R}^2}\left(\mathcal{K}(x,y)u(y) + \partial_\varphi \mathcal{K}(x,y)\partial_\varphi u(y)\right)\rho_j(|y|)\mathcal{W}(y)dy$$

There are vertical spanning of $\{e_n\}_{n\in\mathbb{Z}}$is represented by$u$.
By the specification for the prevailed Toeplitz operator with$\mathcal{W}^2$, it is evident that all signs$\rho_j \in L^\infty(0,\infty)$. Additionally, T_{ρ_j} is to be a multiplier for$\mathcal{W}^2$.

$$T_{\rho_j}\left(\sum a_n e_n\right) = \sum \gamma_n a_n e_n$$

Where

$$\gamma_n = \gamma_{-n} = \frac{2\pi(n^2+1)}{\beta_n^2}\alpha_n \tag{8}$$

And

$$\alpha_n = \int_0^\infty J_{\rho_j}^2(r)\rho_j(r)\mathcal{W}(r)dr. \tag{9}$$

That is evident from a notation on $x = re^{i\vartheta} = r(cos\vartheta + isin\vartheta)$

$$T_{\rho_j}\left(\frac{F_m}{\beta_m}\right)(x) = \int_0^\infty\int_0^{2\pi}\left(\sum_n \frac{J_n(r)J_n(\mathcal{S})e^{in(\vartheta-\varphi)}}{\beta_n^2}\right)\frac{J_m(\mathcal{S})e^{im\varphi}}{\beta_m}d\varphi\rho_j(\mathcal{S})\mathcal{W}(\mathcal{S})d\mathcal{S}$$

$$+\int_0^\infty\int_0^{2\pi}\partial_\varphi\left(\sum_n \frac{J_n(r)J_n(\mathcal{S})e^{in(\vartheta-\varphi)}}{\beta_n^2}\right)\partial_\varphi\left(\frac{J_m(\mathcal{S})e^{im\varphi}}{\beta_m}\right)d\varphi\rho_j(\mathcal{S})\mathcal{W}(\mathcal{S})d\mathcal{S}$$

$$= \left(\frac{2\pi(1+m^2)}{\beta_m^2}\int_0^\infty J_m^2(\mathcal{S})\rho_j(\mathcal{S})\mathcal{W}(\mathcal{S})d\mathcal{S}\right)\frac{F_m}{\beta_m}(x) = \gamma_m\frac{F_m}{\beta_m}(x).$$

On top of that, T_{ρ_j} is restricted if and only if $\{\gamma_n\}_n$ is restricted.
T_{ρ_j} Is bound quickly when $\{\gamma_n\}_n$ has bounds, which means that it is provided in

$$\int_0^\infty J_n^2(r)T_{\rho_j}(r)\mathcal{W}(r)dr \leq \frac{C}{n^2+1} \tag{10}$$

Specifically, while T_{ρ_j} has a limit, then may find for $\rho_j \in L^1([0,\infty),\mathcal{W}(r))$ by using Equation (7). Although Remark 9 shows that, the opposite cannot be correct.

Definition 2: ρ_j Is referred to as a Carlson sign when it is

$$\int_{\mathbb{R}^2}\left(|u(x)|^2 + \left|\partial_\varphi u(x)\right|^2\right)\rho_j(x)\mathcal{W}(x)dx \leq C\int_{\mathbb{R}^2}\left(|u(x)|^2 + \left|\partial_\varphi u(x)\right|^2\right)\mathcal{W}(x)dx$$

That any$u \in \mathcal{W}^2$, that can be, if the identity,$\mathcal{W}^2 \to \mathcal{W}_{\rho_j}^2$, is continuous, when $\mathcal{W}_{\rho_j}^2$. Hence, using the concept of$\mathcal{W}^2$, A Measurement $\mathcal{W}(x)dx$ is substituted w is$\rho_j(x)\mathcal{W}(x)dx$.
Using the operator's notification$T_{\rho_j}^{1/2}$, it can be a densely defined if$T_{\rho_j}^{1/2}u = \sum_n \gamma_{\rho_j}^{1/2} a_n e_n$, suppose that the linear form of the range of $\{e_n\}_{n\in\mathbb{Z}}$ is$u = \sum_{n\in\mathbb{Z}} a_n e_n$. So that$u$,

$$\|u\|_{\mathcal{W}_{\rho_j}^2}^2 = \int_{\mathbb{R}^2} \left(|u(x)|^2 + \left|\partial_\varphi u(x)\right|^2\right) \rho_j(x)\mathcal{W}(x)dx$$

$$= \int_0^\infty \int_0^{2\pi} \left|\sum_n \frac{a_n J_n(\mathcal{S}) e^{in\varphi}}{\beta_m}\right|^2 d\varphi \mathcal{W}(\mathcal{S})\rho_j(x)d\mathcal{S} + \int_0^\infty \int_0^{2\pi} \left|\sum_n \frac{a_n in J_n(\mathcal{S}) e^{in\varphi}}{\beta_m}\right|^2 d\varphi \mathcal{W}(\mathcal{S})\rho_j(x)d\mathcal{S}$$

$$= 2\pi \int_0^\infty \sum_n (n^2+1) \frac{|a_n|^2 J_n^2(\mathcal{S})}{\beta_n^2} \mathcal{W}(\mathcal{S})\rho_j(x)d\mathcal{S} = 2\pi \sum_n \gamma_n \frac{|a_n|^2}{\beta_n^2} = C \left\|T_{\rho_j}^{1/2}u\right\|_{\mathcal{W}^2}.$$

Proposition 3: Using the equivalency arguments listed below
(a) ρ_j Another Carlson sign
(b) T_{ρ_j} This has boundaries.
Now need an estimate of all Basel processes in for the finish, and the estimates can be summed up in the following components.

Lemma 4: Assume$a > 1$, $\mu \geq 0$ and $\varepsilon \geq 1$. For example, when a constant C relies on $(1+\varepsilon)$ with$a$,

$$\frac{1}{C}\mu^{\frac{1}{3}-\frac{1+\varepsilon}{3}} \sum_{j=0}^{M-1} 2^{j\left(1-\frac{1+\varepsilon}{4}\right)} \leq \int_{\frac{\mu}{a}}^{a\mu} \left|J_\mu(r)\right|^{1+\varepsilon} dr \leq C\mu^{\frac{1}{3}-\frac{1+\varepsilon}{3}} \sum_{j=0}^{M-1} 2^{j\left(1-\frac{1+\varepsilon}{4}\right)}$$

When$\mu^{\frac{2}{3}} \sim 2^M$.

Lemma5: If$\mu \geq 1/2$, whether there is a general constant$A > 0$, so that
(a) For$r \geq \mu + \mu^{1/3}$, has been,

$$J_\mu(r) = \frac{1}{\sqrt{2\pi}} \frac{\cos\vartheta(\mu,r)}{(r^2-\mu^2)^{\frac{1}{4}}} + h(\mu,r) \tag{11}$$

And

$$\vartheta(\mu,r) = (r^2-\mu^2)^{\frac{1}{2}} - \mu\, ar\cos\frac{\mu}{r} - \frac{\pi}{4} \tag{12}$$

Also

$$|h(\mu,r)| \leq \begin{cases} A\left(\frac{\mu^2}{(r^2-\mu^2)^{\frac{7}{4}}} + \frac{1}{r}\right) & \text{if } \mu + \mu^{1/3} \leq r \leq 2\mu \\ \frac{A}{r} & \text{if } \quad r \geq 2\mu. \end{cases} \tag{13}$$

(b) $\mu^{1/3}|J_\mu(r)|\sim A$ if $\mu-\mu^{1/3}<r<\mu+\mu^{1/3}$.

(c) A value for α_0 can be chosen independently, of$0<\alpha_0<1/2$, describe thist_0, and then we may use equation μ to define it as

$$\alpha_0=\mu-t_0\mu^{1/3}, \text{ such as } \frac{\alpha_0^2}{2}\mu^{2/3}\le t_0\le \alpha_0^2\mu^{2/3}$$

Where

(a) from μ so that $\alpha=\mu-t\mu^{1/3}$ and $1\le t\le t_0$, we have

$$|J_\mu(\mu\sec h\alpha)|\le A\frac{e^{-\mu(\alpha-\tanh\alpha)}}{\mu^{1/3}t^{1/4}},$$

(b) From $1\le\mu$ so that $\alpha\le\mu$ as α_0 then

$$|J_\mu(\mu\sec h\alpha)|\le A\frac{e^{-\mu(\alpha-\tanh\alpha)}}{\mu^{1/2}}.$$

From $r\ge 2\mu$

$$J_\mu(r)=\sqrt{\frac{2}{\pi r}}\left(\cos\left(r-\frac{\pi\mu}{2}-\frac{\pi}{4}\right)+0\left(\frac{1}{r}\right)\right)$$

This a uniformly inμ.

Proof: Lemma (5) shows this in part 1 by following the steps in (13), (14), part 3 in (16), and parts 2 and 4 in (15).

Lemma 6: If$1\le r\le\mu-\mu^{1/3}$, so that$\phi(r)=\alpha(r)-\tanh\alpha(r)$, where $\alpha(r)$ is provided at a formula$\mu sech\alpha(r)=r$. For example,

(a) ϕ the function should be declining as $\phi'(r)=-\frac{(\mu^2-r^2)^{1/2}}{\mu^r}$

(b)$\phi(\mu\,\text{sech as}\,\alpha_0):=\beta_0>0$,

(c) $\phi(1)\le\log(2\mu)-\frac{\sqrt{\mu^2-1}}{\mu}$

Theorem 7: States that for every constant$\mu_0>0$, if ρ_j is a not negative function $L^1([0,\infty),\mathcal{W}(r)dr)$

$$\sup_{\mu>\mu_0}\int_\mu^\infty\frac{\mu^2}{\mu^{2/3}+r^{1/2}(r-\mu)^{1/2}}\rho_j(r)\mathcal{W}(r)dr<\infty. \quad (14)$$

This indicates thatρ_j, also known as a Carlson sign in (14), accepts a positive value in place ofμ_0. In accordance with Remark 12 and Proposition 13, the aforementioned requirement is nearly essential. The weakness in theorem 7 is reflected in the symbols ρ_j from the integrals indicated in Equation (8) to the use ofμ, which was proved when the propositions were shown in Carlson. This open question has the potential to alter the record in proposition 13.

Lemmas 5, 6, and 7 can be used to control the dependency on a parameter. The steep asymptote of the Bessel functions can be used J_μ to demonstrate these theorems.

It is possible to examine the compactness in the Toeplitz operators. For the purpose of identifying a shift in a type of tearing that is dedicated to the analysis of elements that are part of the Schatten of classes$\mathcal{S}_{1+\varepsilon}$, a compactness of T_{ρ_j} is comparable to matching forces involving$\mathcal{W}^2 \to \mathcal{W}^2_{\rho_j}$.

The compactness of the Toeplitz operators can be analysed. A nonnegative radial function can be found using ρ_j from this work. When$C_1, A \le B \le C_2 A$, it can write $A \sim B$ for two non-negative numbers, A and B, which are positive constantsC_1, C_2.

Proof: Based on the prerequisites in Equation (14) that μ_0 has autonomous, $\rho_j \in L^1([0,\infty), \mathcal{W}(r)dr)$ is given. In Equation (14), let $\rho_j \in L^1([0,\infty), \mathcal{W}(r)dr)$ fulfill. Specified as

$$H(r,\mu) = \frac{\mu^2}{\mu^{2/3} + r^{1/2}|r-\mu|^{1/2}}.$$

That is have$H(r,\mu) \sim \widetilde{H}(r,\mu)$, when

$$\widetilde{H}(r,\mu) = \begin{cases} \mu^{4/3} & r \in \left[\mu - \mu^{1/3}, \mu + \mu^{1/3}\right] \\ \dfrac{\mu^2}{\mu^{1/2}(r-\mu)^{1/2}} & r > \mu + \mu^{1/3} \end{cases}$$

Moreover, consistently to $r \ge \mu - \mu^{1/3}$ and$\mu > \mu_0$.

Note: Equation (14) indicates that there is $\mu_1 > 0$ so that

$$\sup_{\mu > \mu_1} \int_{\mu - \mu^{1/3}}^{\infty} \widetilde{H}(r,\mu)\, \rho_j(r) \mathcal{W}(r) dr < \infty.$$

In the event that we place μ_1 larger, therefore $\mu - \mu^{1/3} \le r \le \mu$ when$\mu \ge \mu_1$, we have$-\mu^{1/3} \sim r \sim \mu$. So that

$$\widetilde{H}(r,\mu) = \mu^{4/3} \le CH\left(r, \mu - \mu^{1/3}\right)$$

Also

$$\sup_{\mu > \mu_1} \int_{\mu - \mu^{1/3}}^{\infty} \widetilde{H}(r,\mu)\, \rho_j(r) \mathcal{W}(r) dr \le C \sup_{\mu > \mu_1} \int_{\mu - \mu^{1/3}}^{\infty} H\left(r, \mu - \mu^{1/3}\right) \rho_j(r) \mathcal{W}(r) dr < \infty.$$

When used with Equation (14), it indicates because

$$\sup_{\mu \ge 0} \mu^2 \int_{\mu + \mu^{1/3}}^{\infty} \frac{\rho_j(r)}{r^{1/2}(r-\mu)^{1/2}} \mathcal{W}(r) dr < \infty$$

and

$$\sup_{\mu \ge 0} \mu^{4/3} \int_{\mu - \mu^{1/3}}^{\mu + \mu^{1/3}} \rho_j(r)\, \mathcal{W}(r) dr < \infty. \tag{15}$$

The series $\{\gamma_n\}_n$ will be able to demonstrate its boundedness while corresponding toρ_j. We deconstruct this at the end for$n \ge 1$

$$\int_0^{\infty} J_n^2(r)\rho_j(r)\mathcal{W}(r)dr = \int_0^{n\,\mathrm{sech}\,\alpha_0} + \int_{n\,\mathrm{sech}\,\alpha_0}^{n-n^{1/3}} + \int_{n-n^{1/3}}^{n+n^{1/3}} + \int_{n+n^{1/3}}^{2n} + \int_{2n}^{\infty} = \sum_{i=1}^{5} J_i$$

Any integration can be estimated according to Lemma 6. It can$|J_\mu(r)| \le C/n$, where$r \in [0, n\,\mathrm{sech}\,\alpha_0)$ similar to

$$\mathcal{J}_1 \le \frac{C}{n^2}\int_0^{\infty} \rho_j(r)\mathcal{W}(r)dr \le \frac{C}{n^2}.$$

Into$\mathcal{J}_3$, that can be $|J_n(r)| \sim n^{-1/3}$ in $[n - n^{1/3}]$, that

$$\mathcal{J}_3 \le \frac{C}{n^{2/3}}\int_{n-n^{1/3}}^{n+n^{1/3}} \rho_j(r)\mathcal{W}(r)dr \le \frac{C}{n^2}\,.$$

To consider$\mathcal{J}_4$, we can know that $|J_n(r)| \le \frac{C}{r^{1/4}(r-n)^{1/4}}$ if$r \in \left[n + n^{1/3}, 2n\right]$, this means that$\mathcal{J}_4 \le \frac{C}{n^2}$. For$\mathcal{J}_5$, which could$|J_\mu(r)| \le Cr^{-1/2}$, so that

$$\int_{2n}^{\infty} J_n^2(r)\rho_j(r)\frac{dr}{r^2} \le C\int_{2n}^{\infty} \frac{r^{1/2}(r-n)^{1/2}}{rr^{1/2}(r-n)^{1/2}}\rho_j(r)\frac{dr}{r^2} \le C\int_{2n}^{\infty} \frac{\rho_j(r)}{r^{1/2}(r-n)^{1/2}}\frac{dr}{r^2} \le \frac{C}{n^2}.$$

Finally, we partition it and determine$\mathcal{J}_2$, $\left[n\,\mathrm{sech}\,\alpha_0, n - n^{1/3}\right] \subset \cup_0^M I_j$ using $I_j = \left\{r = n - \mathcal{S}n^{1/3} : 2^j < \mathcal{S} \le 2^{j+1}\right\}$ As well $M = \left[log_2(1 - \mathrm{sech}\,\alpha_0)n^{2/3}\right]$ at every I_j

$$|J_n(r)| \le \frac{e^{-n(\alpha(r)-\tanh(r))}}{2^{j/4}n^{1/3}}$$

When $r = n\,\mathrm{sech}\,\alpha$
Then:

$$\mathcal{J}_2 = \sum_j \int_{I_j} J_n^2(r)\rho_j(r)\frac{dr}{r^2} \le \sum_j \frac{C}{2^{j/2}n^{2/3}}\int_{.} e^{-2n(\alpha(r)-\tanh(r))}\rho_j(r)\frac{dr}{r^2}$$

$$\le \sum_j \frac{C}{2^{j/2}n^{2/3}}\sup_{I_j} e^{-2n(\alpha(r)-\tanh\alpha(r))}\int_{.} \rho_j(r)\frac{dr}{r^2}$$

The concept is only

$$\psi(\mu) = \mu - \mu^{1/3} \text{ And } \varphi(\mu) = \mu + \mu^{1/3} \tag{16}$$

This is a purpose φ^{-1} at$[0, \infty[$. Enter μ_j as $(\mu_j) = n - 2^{j+1}n^{1/3}$. So that $(\mu_j) \sim \mu_j \sim n$

$$\int_{I_j} \rho_j(r)\mathcal{W}(r)dr = \int_{I_j} \frac{r^{1/2}(r-\mu_j)^{1/2}}{r^{1/2}(r-\mu_j)^{1/2}}\rho_j(r)\mathcal{W}(r)dr$$

$$\leq \int_{I_j} \frac{r^{\frac{1}{2}}\left(r-\varphi(\mu_j)\right)^{\frac{1}{2}}}{r^{\frac{1}{2}}(r-\mu_j)^{\frac{1}{2}}}\rho_j(r)\mathcal{W}(r)dr + \int_{I_j} \frac{r^{1/2}\left(\varphi(\mu_j)-\mu_j\right)^{1/2}}{r^{1/2}(r-\mu_j)^{1/2}}\rho_j(r)\mathcal{W}(r)dr = L_1 + L_2$$

Where$L_1 \leq C2^{j/2}n^{2/3}\int_{\varphi(\mu_j)} \frac{\rho_j(r)}{r^{1/2}(r-\mu_j)^{1/2}}\mathcal{W}(r)dr \leq C2^{j/2}n^{2/3-2},\left(\varphi(\mu_j)-\mu_j\right)^{1/2\sim n^{1/6}}$, and $L_2 \leq Cn^{2/3-2}$

Beginning with the final Lemma7, an $\sup_{I_j} e^{-2n(\alpha(r)-\tanh\alpha(r))} \leq exp\left(-2^{2j/3}\right)$ so we have

$$\mathcal{J}_2 \leq \frac{C}{n^2}\sum_{I_j}\sup_{I_j} e^{-2n(\alpha(r)-\tanh\alpha(r))} \leq \frac{C}{n^2}$$

Example 8: Every purpose $\rho_j \in L^1([0,\infty),\mathcal{W}(r)dr)$ similar to $\rho_j(r)$ stays the same if $r > r_0$ was a sign for Carlson as$\mu > r_0$,

$$\int_{\mu}^{\infty}\widetilde{H}(r,\mu)\,\rho_j(r)\mathcal{W}(r)dr \leq C\int_{\mu}^{\infty}\widetilde{H}(r,\mu)\frac{dr}{r^2} \leq C\left(\mu^{-1/3}+\frac{1}{\mu}\int_{\mu+\mu^{1/3}}^{\infty}\frac{dr}{(r-\mu)^{1/2}}\right) \leq C$$

Particularly while $\widetilde{H}$ is specified in the Theorem 7 proof

$$\rho_j(r) = \frac{1}{(r-a)^{\alpha}}\chi_{(a,\infty)}(r)\,,a>0\,,0\leq\alpha<1$$

As the Carlson, sign infinite
In actuality that

$$\rho_j(r) \leq \frac{1}{(r-a)^{\alpha}}\chi_{(\alpha,a+1)}(r) + \chi_{(a+1,\infty)}(r)$$

Remark 9:

(a) Set $\rho_j \geq 0$ to define the order of λ_n to be

$$\lambda_n = \inf\left\{\rho_j(r): r\in\left[\frac{n}{\gamma},\gamma n\right]\right\},$$

For example, $\gamma > 1$ is infinite. In such a case, is ρ_j not a Carleson sign? As soon as

$$n^2\int_0^{\infty}J_n^2(r)\rho_j(r)\mathcal{W}(r)dr \geq n^2\int_{n/\gamma}^{\gamma n}J_n^2(r)\rho_j(r)\mathcal{W}(r)dr \sim\lambda_n\int_{n/\gamma}^{\gamma n}J_n^2(r)dr \sim\lambda_n$$

In Theorem 5 $\int_{n/\gamma}^{\gamma n}J_n^2(r)dr$ indicates that the constant of the Carlson metric,ρ_j, does not satisfy the following property

$$\lim_{r\to\infty}\frac{\rho_j(r)}{\log r} \geq 0$$

(b) As we observed, have ρ_j similar to

$$\left\{\int_n^{2n} \frac{\rho_j(r)}{r} dr\right\}_{n\in\mathbb{N}}$$

Such ρ_j is an infinite sequence that is not a Carlson sign. In fact, according to Leama6, if$r > 2(n+1)$, so

$$J_n^2(r) + J_{n+1}^2(r) \sim \frac{2}{\pi r}\left(\cos^2\left(r - \frac{\pi n}{2} - \frac{\pi}{4}\right) + \cos^2\left(r - \frac{\pi(n+1)}{2} - \frac{\pi}{4}\right) + 0\left(\frac{1}{r}\right)\right) = \frac{2}{\pi r}\left(1 + 0\left(\frac{1}{r}\right)\right)$$

Hance

$$n^2 \int_0^{\infty} \left(J^2_{\left[\frac{n}{2}\right]}(r) + J^2_{\left[\frac{n}{2}\right]+1}(r)\right)\rho_j(r)\mathcal{W}(r)dr$$

$$\geq n^2 \int_n^{2n} \left(J^2_{\left[\frac{n}{2}\right]}(r) + J^2_{\left[\frac{n}{2}\right]+1}(r)\right)\rho_j(r)\mathcal{W}(r)dr \sim \int_n^{2n} \frac{\rho_j(r)}{r} dr$$

(c) By employing radial signs at position of function ρ_j to show a positive Basle normal v in$[0,\infty)$, the specification about all Toeplitz operators is developed, allowing as

$$T_v u(x) = \int_{\mathbb{R}^2} \left(\mathcal{K}(x,y)u(y) + \partial_\varphi \mathcal{K}(x,y)\partial_\varphi u(y)\right)\mathcal{W}(\mathcal{S})dv(\mathcal{S})d\varphi$$

This $T_v$is the diagonal operator dial (γ_n) and is fundamentally linked to $\{e_n\}$ for$\mathcal{W}^2$, and$\gamma_n = \frac{2\pi(1+n^2)}{\beta_n^2} J_n^2(a)\mathcal{W}(a)$. In equation (30), $a \geq a_0 > 0$is then uniformly bound by$\|T_{\delta_a}\|$.

Example 10: Looking at these examples, one can conclude whether each value found in Carlson's signs is suitable for question. The contrary is demonstrated in the example that follows. Consequently, thus Challenging at determine thus ρ_j specifies a Carlson signs through,

$$\rho_j(r) = \sum_{j=1}^{\infty} \frac{2^{j/2}}{(r-2^j)^{\frac{1}{2}}} \chi M_j(r)$$

When$M_j = \left(2^j + 2^{\frac{j}{3}}, 2^j + 2^{1+\frac{j}{3}}\right)$.

We go over the following recommendations on the significance of condition (8).

Proposition 11: Assume that Equation (15) obtains for a Carlson symbolρ_j

$$\mu^2 \int_{2\mu}^{\infty} \frac{\rho_j(r)}{r^{1/2}(r-\mu)^{1/2}} \mathcal{W}(r)dr \leq C_{\mu_0} \tag{17}$$

Where$\mu > \mu_0$.

Proof: Assuming that ρ_j is a Carleson sign, as previously stated, for$r > 2(n+1)$, we obtain

$$J_n^2(r) + J_{n+1}^2(r) \sim \frac{2}{\pi r}\left(1 + 0\left(\frac{1}{r^2}\right)\right)$$

Then

$$\int_{2(n+1)}^{\infty} \frac{\rho_j(r)}{r^3} dr \le C \int_{2(n+1)}^{\infty} \frac{J_n^2(r) + J_{n+1}^2(r)}{r^2} \rho_j(r)\, dr \le C(\alpha_n + \alpha_{n+1}) \le \frac{C}{n^2}$$

That demonstrates it.

$$\int_{2\mu}^{\infty} \frac{\rho_j(r)}{r^3} dr \le \frac{C}{\mu^2}$$

So

$$\mu^2 \int_{2\mu}^{\infty} \frac{\rho_j(r)}{r^{1/2}(r-\mu)^{1/2}} \mathcal{W}(r) dr \le \mu^2 \int_{2\mu}^{\infty} \frac{\rho_j(r)}{r^3} dr \le C$$

Moreover, (17) holds.

If$n \in \mathbb{N}$, Therefore, Lemma 6 is useful for determination.

$$\int_{n-n^{1/3}}^{n+n^{1/3}} \rho_j(r)\mathcal{W}(r) dr \sim n^{2/3} \int_{n-n^{1/3}}^{n+n^{1/3}} J_n^2(r)\rho_j(r)\mathcal{W}(r) dr \le \frac{C}{n^{4/3}}$$

Where r be large, and then Equation (15) is proven. Take$\mu = n + \alpha$, in which $0 < \alpha < 1$ and$n \in \mathbb{N}$. In Equation (16), enter $\varphi(\mu)$ and$\psi(\mu)$.
Next, Equation (15), which is derived through the earlier state at that moment

$$[\psi(\mu), \varphi(\mu)] \subset [\psi(n), \varphi(n)] \cup [\psi(n+1), \varphi(n+1)]. \tag{18}$$

Remark 12: Based on Proposition 11 and presuming that ρ_j fulfills that

$$\sup_{\mu \ge \mu_0} \frac{1}{\mu^{1/2}} \int_{\mu+\mu^{1/3}}^{2\mu} \frac{\rho_j(r)}{(r-\mu)^{1/2}} dr \le C \tag{19}$$

Therefore, ρ_j is a Carleson sign if and only if it is true according to Equation (8)? As a result, the radial functions ρ_j are found in such a straightforward manner that

$$\sup_{\Omega} \frac{1}{b} \int_a^{a+b} \rho_j(r) dr < C \tag{20}$$

Finally,

$$\Omega = \{\mu, a, b: \mu \geq \mu_0\, , \mu + \mu^{1/3} \leq a \leq 2\mu\, , \mu^{1/3} \leq b \leq \mu\}$$

Equation (19) is made using $\mu_0 > 0$ and C as an absolute constant.
Particularly if it is a fixed $b > 0$

$$\sup_{a>1} \frac{1}{b} \int_a^{a+b} \rho_j(r)dr < \infty, \tag{21}$$

In that case, ρ_j satisfies Equation (20). In this case, we may examine how much condition (19) is necessary forρ_j, a Carleson sign, to be (8). We are able to demonstrate the following outcome:

$$\frac{1}{\mu^{1/2}} \int_{\mu+\mu^{1/3}}^{2\mu} \frac{\rho_j(r)}{(r-\mu)^{1/2}} dr$$

Hence, if ρ_j is a Carleson sign, then it is a query comeμ.
Lemma13: Letμ, while $M \in \mathbb{N}$ so this$2^{M-1} \leq 2\mu \leq 2^M$,

$$(\mu + 2\mu^{1/3}, 2\mu) \subset \bigcup_{j=1}^{M-1} [\mu + 2^j \mu^{1/3}, \mu + 2^{j+1}\mu^{1/3}]$$

As a result of $r \in (\mu + 2^j\mu^{1/3}, \mu + 2^{j+1}\mu^{1/3})$ and $j \in \{1,2, \dots, M-1\}$ it has the potential to

$$\frac{2}{\sqrt{6}} \leq \left|\vartheta(\mu, r) - \vartheta\left(\mu + \frac{\mu^{1/3}}{2^{j/2}}, r\right)\right| \leq 2\sqrt{2} \tag{22}$$

Therefore

$$\cos 2\sqrt{2} \ \leq \cos\left|\vartheta(\mu, r) - \vartheta\left(\mu + \frac{\mu^{1/3}}{2^{j/2}}, r\right)\right| \leq \cos\frac{2}{\sqrt{6}} \tag{23}$$

Proof: If $f(t) = \vartheta(t, r) = (r^2 - t)^{\frac{1}{2}} - t\cos^{-1}\frac{t}{r} - \frac{\pi}{4}\, , r > t$
So that$f'(t) = -\cos^{-1}\frac{t}{r}$.
Then $r \in (\mu + 2^j\mu^{1/3}, \mu + 2^{j+1}\mu^{1/3})$ along with a few of $t \in \left(\mu, \mu + \frac{\mu^{1/3}}{2^{j/2}}\right)$ then

$$\frac{\mu^{1/3}}{2^{j/2}}\cos^{-1}\frac{\mu + \frac{\mu^{1/3}}{2^{j/2}}}{\mu + 2^j\mu} \leq \left|\vartheta(\mu, r) - \vartheta\left(\mu + \frac{\mu^{1/3}}{2^{j/2}}, r\right)\right| = \frac{\mu^{1/3}}{2^{j/2}}\cos^{-1}\frac{t}{r} \leq \frac{\mu^{1/3}}{2^{j/2}}\cos^{-1}\frac{\mu}{\mu + 2^{j+1}\mu}$$

Then $\cos^{-1} x = \int_x^1 \frac{1}{\sqrt{1-t^2}} dt$
Then we have

$$\frac{\mu^{1/3}}{2^{j/2}}\cos^{-1}\frac{\mu}{\mu + 2^{j+1}\mu} \leq \frac{\mu^{1/3}}{2^{j/2}} \int_{\frac{\mu}{\mu+2^{j+1}\mu}}^{1} (1-t^2)^{-1/2}\, dt \leq 2\frac{\mu^{1/3}}{2^{j/2}}\left(1 - \frac{\mu}{\mu + 2^{j+1}\mu}\right)^{1/2} \leq 2\sqrt{2}$$

In a similar way $\frac{\mu^{1/3}}{2^{j/2}}\cos^{-1}\frac{\mu+\frac{\mu}{2^{j/2}}}{\mu+2^{j/2}\mu}\geq\frac{2}{\sqrt{6}}$

Proposition 14: Assume that μ_0 exists since ρ_j is a Carlson sign.

$$\sup_{\mu\geq\mu_0}\frac{1}{\mu^{1/2}log\mu}\int_{\mu+\mu^{1/3}}^{2\mu}\frac{\rho_j(r)}{(r-\mu)^{1/2}}dr\leq C \tag{24}$$

The universal constant,C, exists.A careful examination of the function $\vartheta(\mu,r)$ at lemma 13 is necessary to verify proposition 14 when a transition occurs with a command to the Bessel functions' zero band $\left[\mu+\mu^{1/3},2\mu\right]$. Actually, we can make out of the reality, which

$$\left|J_\mu(r)\right|^2+\left|J_{\mu+\frac{\mu^{1/3}}{2^{j/2}}}(r)\right|^2\geq\frac{C}{(r^2-\mu^2)^{\frac{1}{2}}}$$

When$r\in\left(\mu+2^j\mu^{1/3},\mu+2^{j+1}\mu^{1/3}\right),j\in\{1,2,\dots,M-1\}$, and$2^M\sim\mu^{2/3}$.

Proof: Let $\mu_0>\frac{1}{2}$ while $j_A\in\mathbb{N}$ fulfil then

(a) $\max\left[\frac{16A^2\log\mu_0}{\mu_0},\frac{8A^2}{\mu_0}\right]\leq\frac{1}{8}\frac{1-\cos\frac{2}{\sqrt{6}}}{4\pi}$

(b) $\frac{8A^2}{2^{3j_A}}<\frac{1}{8}\frac{1-\cos\frac{2}{\sqrt{6}}}{4\pi}$

(c) $j_A<\frac{log\mu_0}{100}$

In Equation (14), where an is a constant.

That it suffices to demonstrate that

$$\sup_{\mu\geq\mu_0}\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr\leq C\log\mu \tag{25}$$

In lemma 14's proper number$M$, ρ_j is a completed Carleson sign regardless of when it is?

$$\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\left|J_\mu(r)\right|^2\rho_j(r)dr+\sum_{j=jA}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\left|J_{\mu+\frac{\mu^{1/3}}{2^{j/2}}}(r)\right|^2\rho_j(r)dr\leq C.M\sim C\log\mu \tag{26}$$

Stated differently, this may be accomplished through utilizing

$$|a+b|^2\geq\frac{1}{2}|a|^2-|b|^2$$

They found it.

$$\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\left|J_\mu(r)\right|^2\rho_j(r)dr+\sum_{j=jA}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\left|J_{\mu+\frac{\mu^{1/3}}{2^{j/2}}}(r)\right|^2\rho_j(r)dr$$

$$\geq \frac{1}{4\pi}\sum_{j=jA}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\left(\cos^2\vartheta(\mu,r)+\cos^2\vartheta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right)\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr$$

$$-\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}|h(\mu,r)|^2\,\rho_j(r)dr-\sum_{j=jA}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\left|h\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right|^2\rho_j(r)dr.$$

We can infer from Lemma 6 that in the previous paragraph (b), thus

$$\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}|h(\mu,r)|^2\,\rho_j(r)dr\leq 2A^2\mu^4\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{1}{(r^2-\mu^2)^3}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr+\frac{2A^2}{\mu^2}\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\rho_j(r)dr$$

$$\leq 2A^2\mu^4\sum_{j=j_A}^{M-1}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\frac{1}{(r+\mu)^3(r-\mu)^3}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr+\frac{2A^2}{\mu^2}\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\rho_j(r)dr$$

$$\leq 2A^2\sum_{j=j_A}^{M-1}\frac{1}{2^{3j}}\int_{\mu+2^j\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr+\frac{8A^2}{\mu}\int_{\mu+2^{jA}\mu^{\frac{1}{3}}}^{\mu+2^M\mu^{\frac{1}{3}}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr$$

$$\leq \frac{8A^2}{2^{3j_A}}\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr+\frac{8A^2}{\mu}\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr$$

$$\leq \frac{1}{4}\frac{1-cos\,\frac{2}{\sqrt{6}}}{4\pi}\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^M\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr \qquad (27)$$

In the same way, it can be proven,

$$\sum_{j=jA}^{M-1}\int_{\mu+2^j\mu^{\frac{1}{3}}}^{\mu+2^{j+1}\mu^{1/3}}\left|h\left(\mu+\frac{\mu^{\frac{1}{3}}}{2^{\frac{j}{2}}},r\right)\right|^2\rho_j(r)dr\leq\frac{1}{4}\frac{1-\cos\frac{2}{\sqrt{6}}}{4\pi}\int_{\mu+2^{jA}\mu^{\frac{1}{3}}}^{\mu+2^M\mu^{\frac{1}{3}}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr \qquad (28)$$

From the Lemma 14

$$\cos^2\vartheta(\mu,r)+\cos^2\vartheta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)$$

$$=1+\cos\left(\left|\vartheta(\mu,r)-\vartheta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right|\right).\cos\left(\left|\vartheta(\mu,r)+\vartheta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right|\right)\geq 1-\cos\frac{2}{\sqrt{6}}$$

So that

$$\frac{1}{4\pi}\sum_{j=jA}^{M-1}\int_{\mu+2^{j}\mu^{1/3}}^{\mu+2^{j+1}\mu^{1/3}}\left(\cos^2\vartheta(\mu,r)+\cos^2\vartheta\left(\mu+\frac{\mu^{1/3}}{2^{j/2}},r\right)\right)\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr$$

$$\geq\frac{1-\cos\frac{2}{\sqrt{6}}}{4\pi}\int_{\mu+2^{jA}\mu^{1/3}}^{\mu+2^{M}\mu^{1/3}}\frac{\rho_j(r)}{(r^2-\mu^2)^{1/2}}dr$$

Equation (27) is derived from that and Equation (25).
This allows us to derive Propositions 11 and 13.

Remark 15: Since ρ_j is a Carlesson sign,

$$\sup_{\mu\geq\mu_0}\left\{\int_{(\mu-\infty)-(\mu+\mu^{1/3},2\mu)}\frac{\mu^2}{\mu^{2/3}+r^{1/2}(r-\mu)^{1/3}}\rho_j(r)\mathcal{W}(r)dr\right.$$

$$\left.+\frac{1}{\log\mu}\int_{(\mu+\mu^{1/3},2\mu)}\frac{\mu^2}{\mu^{2/3}+r^{1/2}(r-\mu)^{1/2}}\rho_j(r)\mathcal{W}(r)dr\right\}<\infty.$$

Since Equation (9) defines that with regard to the basis$\{e_n\}$, we recall that each Toeplitz operator T_{ρ_j} is a vertical operator $\{\gamma_n\}_{n\in\mathbb{Z}}$ givenγ_n. A compact if and only iflim$_n\,\gamma_n=0$, then T_{ρ_j} is. We can see that in (15) estimating.

$$\left|J_\mu(r)\right|\leq\frac{r^n}{n!\,2^n}e^{\frac{r^2}{4}}$$

It is evident that for$\rho_j\in L^1\big((0,\infty),\mathcal{W}(r)\big)dr$, this provides an efficient Toeplitz operator when the support is limited. By Proposition 21, there are also compact Toeplitz operators with non-bounded support for their symbols.

Proposition 16: When$T_{\rho_j}\in L^1([0,\infty),\mathcal{W}(r)dr)$. Then T_{ρ_j} is built in $\mathcal{W}^2$ inclusion that $\mathcal{W}^2\to\mathcal{W}_\rho^2$ is compact.

Prof: We can talk about discrepancy based on enough. Letting us assume that $\mathcal{W}^2\to\mathcal{W}_{T_{\rho_j}}^2$was compacted. A subsequence $\{\gamma_{n_\mathcal{K}}\}_{\mathcal{K}\in\mathbb{Z}}$ exists while T_{ρ_j} cannot be compact. For example, $\{\gamma_{n_\mathcal{K}}\}_{\mathcal{K}\in\mathbb{Z}}\geq\varepsilon$ whenever$\varepsilon>0$. It has an approximation value for the relevant sequence $\{e_{n_\mathcal{K}}\}_{\mathcal{K}\in\mathbb{Z}}$ equivalence in$\mathcal{W}_{T_{\rho_j}}^2$. This cannot be done because $\{e_{n_\mathcal{K}}\}_{\mathcal{K}\in\mathbb{Z}}$ was orthogonal with $\mathcal{W}_{\rho_j}^2$ and$\left\|e_{n_\mathcal{K}}\right\|_{W_{\rho_j}^2}=\gamma_n\geq\varepsilon$. Therefore, that $\{\gamma_n\}_{n\in\mathbb{Z}}$ comes together as zero.
For γ_n to be 0, we suppose. The bounded sequence $\{u_n\}_{n\in\mathbb{Z}}$ can be placed in$\mathcal{W}^2$. We may readily see that

$$u_n=\sum_{\mathcal{K}\in\mathbb{Z}}a_{n,\mathcal{K}}e_\mathcal{K}$$

This results in an iterative chain $\{l_n\}_{n\in\mathbb{Z}}$ for $\mathbb{N}$ by aseries in the complex numbers $\{a_{\mathcal{K}}\}_{\mathcal{K}\in\mathbb{Z}}$ in which $\lim_{n\to\infty} a_{l_{n,\mathcal{K}}} = a_{\mathcal{K}}$to every$\mathcal{K}\in\mathbb{Z}$. This is derived from the convergence in$\mathcal{W}^2$. Lemma Fatou's allows us to determine that$\sum_{\mathcal{K}\in\mathbb{Z}}|a_{\mathcal{K}}|^2<\infty$, and we can get it if we put$u=\sum_{\mathcal{K}} a_{\mathcal{K}}e_{\mathcal{K}}\in\mathcal{W}^2$.

$$\left\|u_{l_n}-u\right\|_{\mathcal{W}^2_{\rho_j}}=\sum_{\mathcal{K}\in\mathbb{Z}}\gamma_{\mathcal{K}}\left|a_{n,\mathcal{K}}-a_{\mathcal{K}}\right|^2$$

When it converges to zero, the addition is proven. Is $\mathcal{W}^2\to\mathcal{W}^2_{\rho_j}$ a compact? Our develop a Berezin class translation for the Toeplitz operations on $\mathcal{W}^2$ in order to explain the Hilbert-Smite as Track Category Toeplitz operators.
Put

$$R_x=\frac{1}{(1+|x|^2)^{1/2}}\left(\sum_{n\in\mathbb{Z}}(n^2+1)^{1/2}F_n(x)e_n(x)\right).$$

Our may observe that they have a positive characteristic, meaning

$$c\le\|R_x\|_{\mathcal{W}^2}\le C \tag{29}$$

For any$x\in\mathbb{R}^2$. Actually, from (6) we have

$$\sum_{n\in\mathbb{Z}}(n^2+1)|F_n(x)|^2=\sum_{n\in\mathbb{Z}}(n^2+1)J_n^2(r)=\sum_{n\in\mathbb{Z}}J_n^2(r)=\sum_{n\neq0}n^2J_n^2(r)=1+\sum_{n\neq0}n^2J_n^2(r)$$

According to [15], the summed method

$$J_0(|x-y|)=\sum_{n\in\mathbb{Z}}J_n(r)J_n(\mathcal{S})e^{in(\vartheta-\varphi)}$$

$x=re^{i\vartheta}$,$y=\mathcal{S}e^{i\varphi}$, can got it $x\neq y$

$$\sum_{n\in\mathbb{Z}}n^2J_n(r)J_n(\mathcal{S})e^{in(\vartheta-\varphi)}=\frac{\partial^2}{\partial\vartheta\partial\varphi}J_0(|x-y|)=\frac{J_1(|x-y|)}{|x-y|}x.y+\frac{J_2(|x-y|)}{|x-y|^2}(x.y^{\perp})^2$$

It may proceed according to$\sum_{n\neq0}n^2J_n^2(r)=r^2J_1'(0)=r^2/2$, and

$$\sum_{n\in\mathbb{Z}}(n^2+1)J_n^2(r)\sim(1+r^2) \tag{30}$$

According to (29) earlier.

Definition 17: With Breezing, modify is defined to obtain the sign$\rho_j\in L^1([0,\infty),\mathcal{W}(r)dr)$.

$$B\rho_j(r) = \frac{1}{1+r^2}\left(\sum_{n\in\mathbb{Z}}(n^2+1)\gamma_n J_n^2(r)\right)$$

Our see which whatever $x \in \mathbb{R}^2$ hence,

$$B\rho_j(r) = \left(T_{\rho_j}R_x, R_x\right)$$

Remark 18:
(a) The Breezing transform $B\rho_j$ is limited if ρ_j represents the Carlson sign.
(b) When T_{ρ_j} is assumed to be an efficient operator, so $\lim_{r\to\infty} B\rho_j(r) = 0$

Proof: According to (29), if T_{ρ_j} acts as a limited operator, therefore $\left(T_{\rho_j}R_x, R_x\right)$ has limits on$\mathbb{R}^2$, which indicates (a). Under the assumption that T_{ρ_j} is sufficiently smaller, (γ_n) may come to zero. Considering$\varepsilon > 0$, so given (30)

$$\frac{1}{1+r^2}\left(\sum_{|n|>M}(n^2+1)\gamma_n J_n^2(r)\right) \le \frac{\varepsilon}{1+r^2}\left(\sum_{|n|>M}(n^2+1)J_n^2(r)\right) \le C\varepsilon$$

When the M is sufficiently large. Next,

$$\lim_{r\to\infty} sup B\rho_j(r) = \lim_{r\to\infty} sup\left(\left(\sum_{|n|\le M} + \sum_{|n|>M}\right)(n^2+1)\gamma_n J_n^2(r)\right) \le C\varepsilon$$

Thus$\lim_{r\to\infty} B\rho_j(r) = 0$.
Let us examine our Toeplitz operators that are a part of the Schatten classes$\mathcal{S}_{1+\varepsilon}$. At a Hilbert space, a limited operator T belongs to the Schatten category $\mathcal{S}_{1+\varepsilon}$ where$Tr|T|^P < \infty$. From (17), TrA represents the operator as trace, A so$|T| = \sqrt{T^*T}$. With regard to the basis$\{e_n\}_{n\in\mathbb{Z}}$, $T_{1+\varepsilon}$ a horizontal operator? Therefore, $T_{1+\varepsilon} \in \mathcal{S}_{1+\varepsilon}$is possible if and only if$\{\gamma_n\} \in \ell^P(\mathbb{Z})$. Here,$\|T_{1+\varepsilon}\|_{\mathcal{S}_{1+\varepsilon}} = \|\gamma_n\|_{l_{1+\varepsilon}}$. We may afterward derive that next theorem according to the Hilbert–Schmidt and record type$\mathcal{S}_1$. Operators$\mathcal{S}_2$.

Theorem 19: The statements that follow are equivalent:
(a) $T_{1+\varepsilon} \in \mathcal{S}_1$
(b) $\rho_j \in L^1\left((1+r^2)\mathcal{W}(r)dr\right)$
(c) $B\rho_j \in L^1\left((1+r^2)\mathcal{W}(r)dr\right)$
Additionally, for amounts $\|T_{1+\varepsilon}\|_{\mathcal{S}_1}, \left\|\rho_j\right\|_{L^1\left((1+r^2)\mathcal{W}(r)dr\right)}, \left\|B\rho_j\right\|_{L^1\left((1+r^2)\mathcal{W}(r)dr\right)}$
They can be similar.
Proof: Since (30), it has

$$\sum_{n\in\mathbb{Z}}\gamma_n \sim \sum_{n\in\mathbb{Z}}(n^2+1)\int_0^\infty \rho_j(r)J_n^2(r)\mathcal{W}(r)dr$$

$$=\int_0^\infty \rho_j(r)J_n^2(r)\mathcal{W}(r)dr\sum_{n\in\mathbb{Z}}(n^2+1)J_n^2(r)dr\sim\int_0^\infty \rho_j(r)(1+r^2)\mathcal{W}(r)dr$$

A new perspective is used to prove solutions (a) and (b).

$$\sum_{n\in\mathbb{Z}}\gamma_n \sim \sum_{n\in\mathbb{Z}}\gamma_n\int_0^\infty (n^2+1)J_n^2(r)\mathcal{W}(r)dr=\int_0^\infty B\rho_j(r)J_n^2(1+r^2)\mathcal{W}(r)dr$$

At this point, (a) and (c) are equal.

Theorem 20: Put $T_{1+\varepsilon}\in\mathcal{S}_2$ when not otherwise if$\rho_j B\rho_j\in L^1\big((1+r^2)\mathcal{W}(r)dr\big)$, furthermore $\|T_{1+\varepsilon}\|_{\mathcal{S}_2}$ so that $\big\|\rho_j B\rho_j\big\|^{1/2}_{L^1\left((1+r^2)\mathcal{W}(r)dr\right)}$ is equivalent quantities.

Proof: Then can be fined.

$$\sum_{n\in\mathbb{Z}}\gamma_n^2 \sim \sum_{n\in\mathbb{Z}}\gamma_n(n^2+1)\int_0^\infty \rho_j(r)J_n^2(r)\mathcal{W}(r)dr$$

$$=\int_0^\infty \sum_{n\in\mathbb{Z}}\gamma_n(n^2+1)J_n^2(r)\rho_j(r)\mathcal{W}(r)dr\sim\int_0^\infty \rho_j(r)B\rho_j(r)(1+r^2)\mathcal{W}(r)dr$$

This is the following section, generally.

Proposition21: Let$\varepsilon\geq 1$. So that

(a) If$\rho_j\in L^{1+\varepsilon}\big((1+r^2)\mathcal{W}(r)dr\big)$, thereafter $T_{1+\varepsilon}\in\mathcal{S}_2$ so that

$$\|T_{1+\varepsilon}\|_{\mathcal{S}_{1+\varepsilon}}\leq C_{1+\varepsilon}\big\|\rho_j\big\|_{L^{1+\varepsilon}\left((1+r^2)\mathcal{W}(r)dr\right)} \tag{31}$$

(b) If$T_{1+\varepsilon}\in\mathcal{S}_{1+\varepsilon}$, thereafter $B\rho_j\in L^{1+\varepsilon}\big((1+r^2)\mathcal{W}(r)dr\big)$ so that

$$\big\|B\rho_j\big\|_{L^{1+\varepsilon}\left((1+r^2)\mathcal{W}(r)dr\right)}\leq C\|T_{1+\varepsilon}\|_{\mathcal{S}_{1+\varepsilon}} \tag{32}$$

Proof: As soon as

$$\int_0^\infty J_n^2(r)\mathcal{W}(r)dr\sim(n^2+1)^{-1}$$

Jensen's inequality and the approximation in Equation (30) give us

$$\sum_{n\in\mathbb{Z}} \gamma_n^{1+\varepsilon} \le C \sum_{n\in\mathbb{Z}} \left((n^2+1) \int_0^\infty \rho_j(r) J_n^2(r) \mathcal{W}(r) dr \right)^{1+\varepsilon}$$

$$\le C \sum_{n\in\mathbb{Z}} \left(\frac{\int_0^\infty \rho_j(r) J_n^2(r) \mathcal{W}(r) dr}{\int_0^\infty J_n^2(r) \mathcal{W}(r) dr} \right)^{1+\varepsilon} \le C \sum_{n\in\mathbb{Z}} \frac{\int_0^\infty {\rho_j}^{1+\varepsilon}(r) J_n^2(r) \mathcal{W}(r) dr}{\int_0^\infty J_n^2(r) \mathcal{W}(r) dr}$$

$$\le C \int_0^\infty \sum_{n\in\mathbb{Z}} (n^2+1) J_n^2(r) {\rho_j}^{1+\varepsilon}(r) \mathcal{W}(r) dr \le C \int_0^\infty {\rho_j}^{1+\varepsilon}(r)(1+r^2)\mathcal{W}(r) dr$$

That is proof (31).

Jensen's inequality once more gives us by Equation (30) because

$$B^{1+\varepsilon}\rho_j(r) \le \frac{1}{1+r^2} \left(\sum_{n\in\mathbb{Z}} \gamma_n^{1+\varepsilon} (n^2+1) J_n^2(r) \right)$$

After that, you can obtain (32) by including that disparity about regards to$(1+r^2)\mathcal{W}(r)dr$. Also, observe that the quantity $(1+r^2)\mathcal{W}(r)dr$ is used to prove whether the Berezin transformed $B\rho_j$ is an integrating operator over$[0,\infty)$.

$$B\rho_j(r) = \int_0^\infty M(r,\mathcal{S})\, \rho_j(\mathcal{S})(1+\mathcal{S}^2)\mathcal{W}(\mathcal{S}) d\mathcal{S}$$

In this instance, its symmetric kernel must be$M(r,\mathcal{S})$.

$$M(r,\mathcal{S}) = \frac{2\pi}{(1+r^2)(1+\mathcal{S}^2)} \sum_{n\in\mathbb{Z}} \frac{(n^2+1)^2}{\beta_n^2} J_n^2(r) J_n^2(\mathcal{S})$$

Corollary 22: For each$1 \le \varepsilon \le \infty$, the Berezin transform $B\rho_j$ must have finite integrated operators on $L^{1+\varepsilon}\big((1+r^2)\mathcal{W}(r)dr\big)$.

Proof: The statement for $\varepsilon \in [0,1]$ can be proved simply by pointing out that M is symmetric. This falls under Theorem 19 with$\varepsilon = 0$. It is finally proven with$\varepsilon = 1$. According to Proposition 21, $T_{\rho_j} \in \mathcal{S}_2$ when$\rho_j \in L^2\big((1+r^2)\mathcal{W}(r)dr\big)$. In the event that (γ_n) denotes a sequence that corresponds toT_{ρ_j}, so Proposition 21 and Theorem 20 afford us

$$\left\|B\rho_j\right\|^2_{L^2\left((1+r^2)\mathcal{W}(r)dr\right)} \le \sum_{n\in\mathbb{Z}} \gamma_n^2 \le C_{1+\varepsilon} \int_0^\infty \rho_j(r) B\rho_j(r)(1+r^2)\mathcal{W}(r)dr \le$$

$$\left\|\rho_j\right\|_{L^2\left((1+r^2)\mathcal{W}(r)dr\right)} \left\|B\rho_j\right\|_{L^2\left((1+r^2)\mathcal{W}(r)dr\right)}$$

That is what we may conclude.

$$\|B\rho_j\|_{L^2\left((1+r^2)w(r)dr\right)} \leq C_{1+\varepsilon}\|\rho_j\|_{L^2\left((1+r^2)w(r)dr\right)}$$

Finally, we use Completion to get the value$\varepsilon \in [0,1]$.

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